ISSN 1062-8738, Bulletin of the Russian Academy of Sciences: Physics, 2017, Vol. 81, No. 1, pp. 80–83. © Allerton Press, Inc., 2017.
Original Russian Text © A.V. Nikolaeva, O.A. Sapozhnikov, 2017, published in Izvestiya Rossiiskoi Akademii Nauk, Seriya Fizicheskaya, 2017, Vol. 81, No. 1, pp. 89–92.

Acoustic Radiation Force of a Quasi-Gaussian Beam Imparted to a Solid Spherical Scatterer in a Fluid

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Abstract—The radiation force generated upon the scattering of a quasi-Gaussian acoustic beam on a homogeneous elastic sphere in a fluid is investigated. It is shown that the force depends nonmonotonically on the ratio between the sphere's diameter and the beam's waist. For a given beam power, the radiation force has its maximum value when the diameters are roughly egual to each other. This is due to the resonant excitation of shear waves on the sphere's surface under the impact of acoustic wave in the surrounding fluid.

DOI: 10.3103/S1062873817010208

INTRODUCTION

The use of ultrasonic waves in medicine and industrial technologies is a rapidly developing area of applied acoustics. Not only traditional ways of exposing a medium to ultrasound by heating or cavitation are used, but more precise nonlinear phenomena as well. One of these is the generation of radiation pressure (the acoustic radiation force). Several years ago, a new way of treating urolithiasis was proposed that consisted of the noninvasive propulsion of small stones from a kidney under the impact of an ultrasonic beam [1, 2]. The possibility of repositioning and levitating small scatterers via the effect of ultrasound radiation pressure has been known for a long time [3]. The radiation force was investigated in a number of works for scatterers whose dimensions were comparable to the wavelength. Different techniques for experimentally measuring the force were proposed in [4, 5].

To optimize the radiation force action on a scatterer, we must have an effective numerical algorithm for accurate and quick calculation of the radiation force that allows us to determine its dependence on the diameter and elastic properties of a scatterer, and on the transverse size, intensity, and frequency of the acoustic beam. Such an algorithm for an elastic spherical scatterer and a quasi-Gaussian acoustic beam is proposed in this work, and the dependence of the radiation force on the ratio of the diameters of the scatterer and the beam is analyzed.

THEORETICAL MODEL

As it is known, the radiation force appears as a result of the scattering of an incident acoustic beam on an investigated body. Calculations of the radiation force are thus based on analyzing the scattering for the acoustic beam. With a quasi-Gaussian beam, this problem can be solved using the results obtained in [6]. Let us briefly consider the calculation algorithm.

The following equation [6] can be used as the solution to the Helmholtz equation for an incident quasi-Gaussian beam:

$$p_i = p_0 \frac{z_d}{2\sinh^2(kz_d)} \times \left[e^{kz_d} \frac{\sin\left(k\sqrt{r_\perp^2 + (z - iz_d)^2}\right)}{\sqrt{r_\perp^2 + (z - iz_d)^2}} - e^{-kz_d} \frac{\sin\left(k\sqrt{r_\perp^2 + (z + iz_d)^2}\right)}{\sqrt{r_\perp^2 + (z + iz_d)^2}} \right], \tag{1}$$

where p_i is the complex amplitude of acoustic pressure in an incident wave, p_0 is the initial wave amplitude on the beam axis, $k = \omega/c$ is the wavenumber, c is the speed of sound in fluid, $z_d = ka_0^2/2$ is the length of the diffraction divergence of beam, a_0 is the beam waist radius, and $r_\perp = \sqrt{x^2 + y^2}$ is the transverse coordinate (the distance from the beam axis). The solution to Eq. (1) is a superposition of two pairs of sources and sinks that ensures a wave does not propagate in the opposite direction and prevents the emergence of singularities and branch points in the solution. Representation (1) describes the beams at an arbitrary degree of their focusing, including a focal waist on the scale of the order diffraction limit ($ka_0 \le 1$). When $kz_d \gg 1$ Eq. (1) becomes the solution for a Gaussian beam.

The quasi-Gaussian beam structure presented by Eq. (1) depends on the numerical value of parameter ka_0. With a fixed value of beam radius a_0 and $ka_0 = 1$ the wave structure barely resembles a directional beam. When $ka_0 = 2$, however, directivity is expressed more clearly, and the divergence of quasi-Gaussian beam is reduced as ka_0 grows.

Since the radiation force appears as a result of the partial transfer of momentum to the scattering object, we must first solve the scattering problem. To do so and subsequently calculate the radiation force, let us consider the more general case of an arbitrary axial-symmetric incident beam. We write the complex amplitude of the acoustic pressure in this beam in the form of expansion by spherical harmonics:

$$p_i = \sum_{n=0}^{\infty} Q_n j_n(kr) P_n(\cos\theta), \tag{2}$$

where r and θ are spherical coordinates, $P_n(\cos\theta)$ denotes Legendre polynomials, and $j_n(x)$ are spherical Bessel functions. For the incident wave described by (2), the scattered wave can be expressed as a superposition of outgoing waves represented by the following series

$$p_s = \sum_{n=0}^{\infty} c_n Q_n h_n^{(1)}(kr) P_n(\cos\theta), \tag{3}$$

where $h_n^{(1)}(kr)$ are spherical Hankel functions, and coefficient c_n characterizes the scattering of the corresponding spherical harmonics. When the scatterer is a isotropic elastic sphere with its center at the origin (which is considered below), the expressions for scattering coefficient c_n have the familiar analytical form presented in [7–10].

Being the vector variable, the radiation force generally has three Cartesian components and can move millimeter objects in arbitrary directions. When both the beam and the scatterer are axially symmetric, we are left with component F_z moving the scatterer along the axis of propagation. It can be expressed by coefficients c_n and Q_n [7, 8]:

$$F_z = \frac{2\pi}{\rho c^2 k^2} \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)(2n+3)} \times \mathrm{Im}\left\{ Q_n Q_{n+1}^* \left(c_n + c_{n+1}^* + 2 c_n c_{n+1}^* \right) \right\}. \tag{4}$$

To apply common expression (4) to a specific type of incident beam, the expansion coefficient is conveniently written as

$$Q_n = p_0 i^n (2n+1) g_n(kz_d). \tag{5}$$

Here, the multiplier $g_n(kz_d)$ considers the difference between the beam and a plane wave (the solution for a plane wave corresponds to when $g_n = 1$). For the quasi-Gaussian beam described by Eq. (1), dependency $g_n(kz_d)$ can be expressed analytically as [6]

$$g_n(x) = \frac{1 - (-1)^n e^{-2x}}{\left(1 - e^{-2x}\right)^2} \mathrm{e}^{-x} \sqrt{2\pi x} I_{n+1/2}(x), \tag{6}$$

where $I_{n+1/2}(x)$ is the Infeld function.

NUMERICAL CALCULATION

Expressions (4)–(6) are used for numerical calculation of the radiation force from a quasi-Gaussian beam incident on a elastic spherical scatterer in a fluid. Our calculations were performed in Fortran. Expression (4) was summed from $n = 0$ to $n = (5–7)\ ka$.

The calculations were performed for two types of elastic spheres. The first scatterer was made of COM (calcium oxalate monohydrate or vevellit), the characteristics of which are closest to one type of kidney stone with density $\rho_* = 2038\ \mathrm{kg/m^3}$ and speed of longitudinal and shear waves $c_l = 4535\ \mathrm{m/s}$, $c_s = 2132\ \mathrm{m/s}$, respectively [11]. The second considered scatterer was a stone made of U-30 (Ultracal; $\rho_* = 1700\ \mathrm{kg/m^3}$, $c_l = 2630\ \mathrm{m/s}$, $c_s = 1330\ \mathrm{m/s}$ which is often used to create phantoms of kidney stones [12].

The dependences of radiation force magnitude $Y_g = F_z c/W$ (where $W = \frac{\pi a_0^2 p_0^2}{4\rho c} \frac{\cosh(kz_d)}{\sinh(kz_d)}$ is the acoustic power of a quasi-Gaussian beam [6]) normalized for the beam acoustic power on parameter ka with fixed ratio of waist radius a_0 to scatterer radius a were plotted for both types of scatterers (Fig. 1). Wavenumber $k = 2\pi f/c$ is a variable parameter proportional to the frequency of the acoustic wave. The magnitude of the radiation force depends nonmonotonically on the scatterer radius, and the behavior of this dependence is associated with the elastic properties of the spherical object's material. The occurance of local minima and maxima is due to resonant oscillations of the scatterer at certain frequencies. Plotting this type of dependence for specific materials allows us to determine the frequencies of the maximum magnitude of radiation force or frequency that must be avoided due to ineffective force generation achieved at one and the same acoustic power.

The dependences of normalized radiation force $Y_g = F_z c/W$ on the radius of the beam waist were also plotted for the considered scatterers using different fixed values of scatterer radius a_0 The calculated dependencies are represented in Fig. 2.

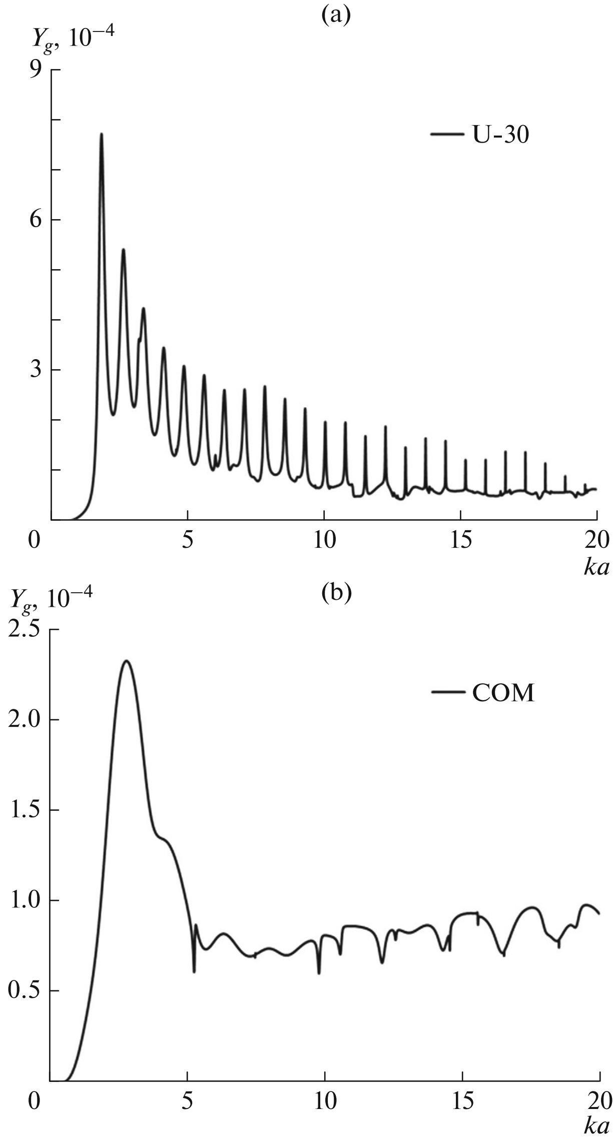

Fig. 1. Dependence of normalized values of the radiation force on parameter *ka* at a fixed ratio of the beam radius to the scatterer radius ($a_0/a = 1$) for two types of materials: (a) U-30 and (b) COM.

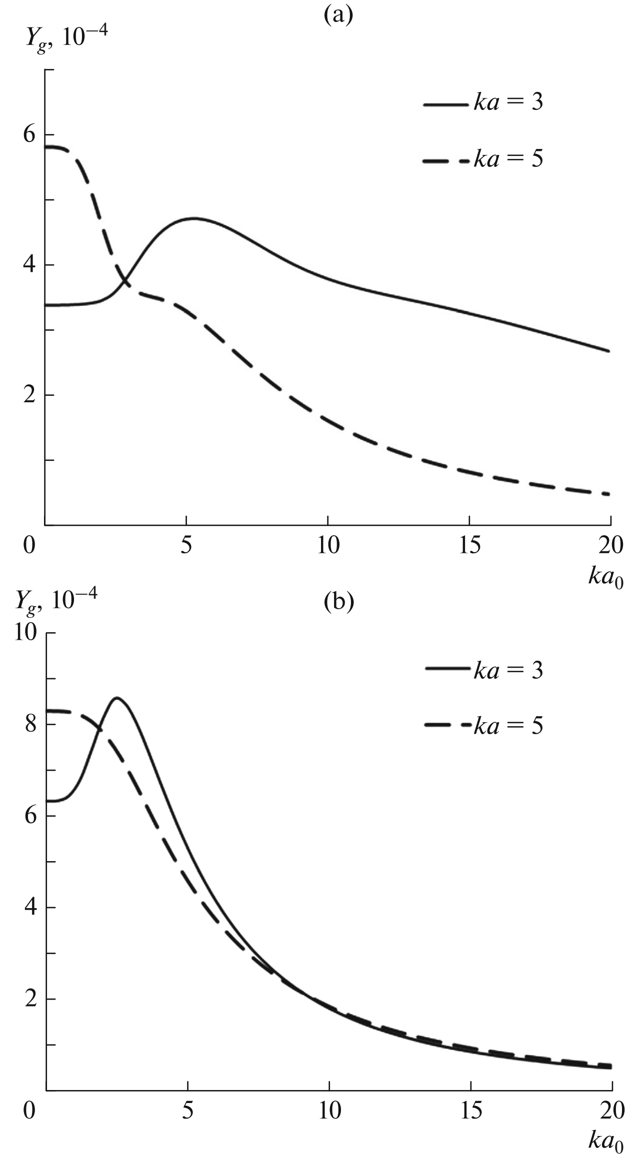

Fig. 2. Dependence of the normalized values of the radiation force on the waist radius of a quasi-Gaussian beam at different fixed values of parameters *ka* (ka = 3, 5; a = 2.5 mm). The diagrams are plotted for two types of elastic spherical scatterers: (a) U-30 and (b) COM.

The curves look counterintuitive. It is obvious to assume that the force falls monotonically as the beam waist radius grows and a rising proportion of the wave energy travels past the scatterer without affecting it. We might therefore expect that the magnitude of radiation force would be greatest if the ultrasonic beam radius were much smaller than the scatterer radius ($a_0 \ll a$), since the beam is fully incident on the sphere and has the greatest force action. However, the dependence for $ka = 3$ presented in Fig. 2 contradicts this assumption. Agreement is seen only in area $a_0 \gg a$ when the beam is significantly wider than the scatterer, which can be explained by the fact that the majority of the beam energy misses the target. At the same time, the maximum value of the radiation force is achieved not when $a_0 \ll a$, but when the beam waist radius is comparable to the scatterer radius ($a_0 \simeq a$) when the beam "wraps around" the sphere. Even though some of this energy does travel past the scatterer, the force acting on it grows relative to when $a_0 \ll a$. With different configurations between a_0 and a, the difference in force compared to a very narrow beam can be as high as 40%. When $ka = 5$, the effect is not observed for either COM or U-30, most likely because at this parameter the characteristic dependence of the radiation force on *ka* lies in one of the points of minimum, and the effect of radiation force is particularly weak (Fig. 1).

DISCUSSION

To understand the reason for this effect, we must refer to an earlier study on the destruction of kidney stones [13]. It was shown there that when a scatterer is exposed to shock pulses the elastic stresses originating in it depend directly on the shear waves generated in the scatterer. The maximum tension is reached when the focal area dimensions of the acoustic pulse slightly exceed the linear dimensions of the stone. The stone is thus most effectively fragmented when the propagating pulse flows past the scatterer. This conclusion is valid for the situation considered in this work. The main reason for the effect of reaching the maximum value of the radiation force when the dimensions of the scatterer and the quasi-Gaussian beam are comparable is that as the ultrasonic beam propagates through the scatterer, the speed of the shear waves generated in it is close to the speed of sound in the fluid. Side areas thus emerge on the scatterer that effectively capture the energy of the beam flowing past the scatterer, facilitating the penetration of wave energy into it and a more efficient transfer of momentum to it.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project nos. 14-02-00426, NIH DK43881, and NSBRI/NASA NCC 9-58.

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Translated by M. Kromin