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Nonlinear Distortion and Attenuation of Intense Acoustic Waves in Lossy Media Obeying a Frequency Power Law

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Abstract—Nonlinear propagation of a periodic wave and a single pulse with a shock front through a lossy medium is studied theoretically. The medium is characterized by the frequency dependence of the attenuation coefficient obeying a power law and by a corresponding dispersion law. The numerical modeling of the problem is performed on the basis of the modified spectral approach. It is found that the exponent of the aforementioned power law essentially affects the efficiency of nonlinear interactions, the distortion of the wave profile, and the absorption of acoustic energy in the nonlinear mode. The stability of the discontinuous structure of a shock front is investigated for different power laws close to a linear one. The possibility of pulsed diagnostics of the parameters of the frequency power law governing the attenuation in the medium by the shape of a single pulse with a shock front after its passage through the medium is considered. © *2000 MAIK "Nauka/Interperiodica".*

The nonlinear effects that accompany the propagation of intense acoustic waves in lossy media have been studied in detail for the case of a classical fluid with a square-law frequency dependence of the attenuation coefficient and for a medium with a single relaxation time [1, 2]. However, in many acoustic media, e.g., biological tissues [3] or sea sediments [4], the frequency dependence of the attenuation coefficient deviates from a square-law one. The theoretical description of nonlinear interactions in such media requires the solution of fairly complicated integro-differential equations, and, therefore, the problems related to this phenomenon have been little investigated. However, these problems are important for many practical applications of intense ultrasound and, specifically, for medical acoustics including hyperthermia, acoustical surgery [3, 5], and extracorporeal lithotripsy [6]. The fundamental aspect of the problem is also of interest, especially, in relation to the study of severely distorted nonlinear disturbances containing steep segments, i.e., shock fronts.

This paper presents a theoretical study of the specific features of the nonlinear propagation of periodic and pulsed disturbances in such media; namely, the effect of the parameters of the power law, which governs the attenuation in the medium, on the evolution of the wave profile, the structure of the shock front, and the wave attenuation. The mathematical model developed below allows for the nonlinear effects, the attenuation, and the sound velocity dispersion. The numerical calculations are based on the modified spectral approach [7, 8] that allows one to describe severely distorted waves containing discontinuities by a small number of harmonics. We select the characteristic parameters of the medium and the initial signals to be close to those used in ultrasound therapy. We also discuss the possibility of a pulsed diagnostics of the parameters of the power law, which governs the frequency dependence of attenuation, by single intense pulses with shock fronts.

To describe the propagation of an acoustic wave of finite amplitude in a medium with an arbitrary attenuation law and an arbitrary dispersion, we use a Burgers-type equation

$$\frac{\partial p}{\partial x} - \frac{\varepsilon}{c_0^3\rho_0} p \frac{\partial p}{\partial \tau} = L(p). \quad (1)$$

Here, p is the acoustic pressure, x is the coordinate of the propagating wave, $\tau = t - x/c_0$ is the time in the moving coordinate system, c_0 is the phase velocity of sound at the characteristic frequency ω_0, ε is the coefficient characterizing the acoustic nonlinearity of the medium, and $L(p)$ is the linear operator responsible for the attenuation and dispersion.

We assume that the frequency dependence of the attenuation coefficient is described by the power law

$$\alpha(\omega) = \alpha_0(\omega/\omega_0)^{\eta}. \quad (2)$$

In this case, it is impossible to rearrange the right-hand member $L(p)$ of equation (1) in a unique way with the causality principle being met. The reason is that the power law (2) cannot be obeyed in the entire infinite frequency range; in particular, at high frequencies, the dependence $\alpha(\omega)$ tends to saturation [9]. The ambiguity of the selection of the operator $L(p)$, which should correspond to attenuation law (2) and meet the causality principle, is related to the possibility of using different models of attenuation at high frequencies. Several different integral forms of $L(p)$ can be found in the literature [4, 10–13]. The solution of the aforementioned integro-differential equations describing the propagation of severely distorted nonlinear waves presents an extremely complicated problem. Even in numerical modeling, the integral form of equations makes it necessary to perform a convolution at every step of the computational scheme, which considerably complicates and slows down the calculations [14]. Approximate analytical solutions can only be obtained for some particular cases [13].

In our study, for describing nonlinear waves in media where the frequency dependence of attenuation is governed by a power law, we use the spectral approach. We consider a system of coupled equations for an infinite number of harmonics; for each harmonic, the corresponding attenuation and dispersion are taken into account.

We assume that, in a broad frequency band, the attenuation obeys the power law (2). According to the causality principle, we can write the Kramers–Kronig-type integral dispersion relations between the attenuation law and the sound velocity dispersion [9, 15]. In the case of a smooth frequency dependence of the attenuation coefficient, we can determine the approximate relationship between the attenuation and dispersion by using the so-called local dispersion relations [15]. From the given attenuation law and the dependences obtained for the sound velocity, we can calculate the characteristics of the nonlinear propagation of waves without any rearrangement of the evolution equation (1).

Now, in equation (1), we pass to dimensionless variables:

$$\frac{\partial V}{\partial z} - NV\frac{\partial V}{\partial \theta} = L'(V), \qquad (3)$$

where $V = p/p_0$ is the acoustic pressure normalized to the characteristic amplitude value p_0; $\theta = \omega_0\tau$ is time in the moving coordinate system; $z = x/x_{at}$ is the wave propagation coordinate normalized to the attenuation length $x_{at} = 1/\alpha_0$; α_0 is the attenuation coefficient at the frequency ω_0; ω_0 is the characteristic frequency of the acoustic signal, where, for a pulse disturbance of duration t_0, the frequency is $\omega_0 = 1/t_0$; $x_{nl} = c_0^3\rho_0/\varepsilon p_0\omega_0$ is the length of the discontinuity formation for a harmonic wave in the absence of attenuation; $N = x_{at}/x_{nl}$ is the dimensionless parameter of nonlinearity; and the operator $L'(V)$ corresponds to the initial operator $L(p)$ expressed through the new variables.

We represent the solution to equation (3) in the form of a Fourier series expansion

$$V(z, \theta) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n(z)\exp(-in\theta). \qquad (4)$$

Then, substituting solution (4) into equation (3), we obtain a system of an infinite number of coupled equations for the Fourier components:

$$\frac{dC_n}{dz} = -\frac{in}{2}N \times\left(2C_0C_n + \sum_{k=1}^{n-1} C_kC_{n-k} + 2\sum_{k=n+1}^{\infty} C_kC_{k-n}^*\right) + iK'(n)C_n - K''(n)C_n. \qquad (5)$$

Here, K' and K'' are the real and imaginary parts of the dimensionless wave number $K(n) = K' + iK''$. They describe the dispersion (K') and the attenuation (K'') of sound, and, in the moving coordinate system, they have the form:

$$K''(n) = \alpha(n\omega_0)/\alpha_0, \quad K'(n) = n\omega_0(1/(c(n\omega_0) - 1/c_0)/\alpha_0. \qquad (6)$$

The frequency dependence of the attenuation coefficient K'' was selected according to the experimental data for biological tissues [3, 16], and the sound velocity dispersion K' was calculated using the local dispersion relations [15]

$$\alpha(\omega) = \frac{\pi\omega^2}{2c_0^2}\frac{dc(\omega)}{d\omega}, \qquad (7)$$

$$\Delta c = c(\omega) - c_0 = \frac{2c_0^2}{\pi}\int_{\omega_0}^{\omega}\frac{\alpha(\omega')}{(\omega')^2}d\omega'. \qquad (8)$$

For the attenuation obeying the power law (2), the sound velocity dispersion calculated by formula (8) has the form

$$\frac{\Delta c}{c_0} = \frac{c(\omega) - c_0}{c_0} = \frac{2c_0\alpha_0}{\pi(\eta - 1)\omega_0}\begin{cases}((\omega/\omega_0)^{\eta-1} - 1), & \eta \neq 1 \\ \ln(\omega/\omega_0), & \eta = 1.\end{cases} \qquad (9)$$

We select the parameters of the power law (2) to be close to the parameters of biological media or the biological tissue phantom 1.3 butanediol [16]. Figure 1a shows the frequency dependences of the attenuation coefficient normalized to its value α_0 at 1 MHz. The curves presented in the figure refer to blood ($\eta = 1.42$,

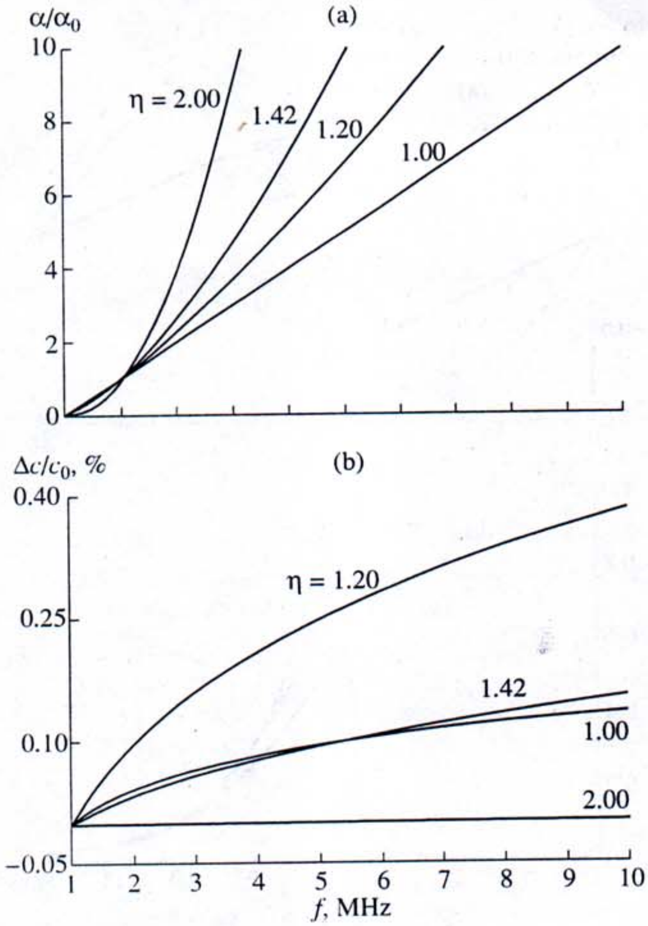

Fig. 1. Frequency dependences of (a) the attenuation coefficient and (b) the sound velocity dispersion for different exponents of the frequency power law governing the attenuation in the medium: $\eta = 2$ (water), 1.42 (blood), 1.2 (liver), and 1 (butanediol).

$\alpha_0 = 0.025$ cm^{-1}, and $c_0 = 1570$ m/s), liver ($\eta = 1.2$, $\alpha_0 = 0.082$ cm^{-1}, and $c_0 = 1600$ m/s) [3], water ($\eta = 2$, $\alpha_0 = 0.0003$ cm^{-1}, and $c_0 = 1500$ m/s), and butanediol ($\eta = 1$, $\alpha_0 = 0.038$ cm^{-1}, and $c_0 = 1546$ m/s). The sound velocity calculated by formula (9) with these data is shown in Fig. 1b. From this figure, one can see that the dispersion characteristics of butanediol and blood are close to each other. The theoretical dispersion curves shown in the figure agree well with the experimental data [3], which testify that the values of the sound velocity dispersion in biological tissues do not exceed 1% in the frequency range 1–10 MHz.

To perform a numerical integration of the system of equations (5), we use a modified spectral approach based on the *a priori* data on the high-frequency asymptotics of the shock-wave spectrum. The main idea of the approach developed in our previous publication [7] lies in the replacement of the exact system of an infinite number of equations (5) by an approximate system of equations for the amplitudes of the first N_{max} harmonics:

$$\frac{dC_n}{dz} = -inN\left\{C_0C_n + \frac{1}{2}\sum_{k=1}^{n-1} C_kC_{n-k} + \sum_{k=n+1}^{N_{max}} C_kC^*_{k-n} + \sum_{k=N+1}^{N_{max}-n} \tilde{C}_kC^*_{k-n} + \left(\frac{A_s}{2\pi}\right)^2\frac{\exp(in\theta_s)}{n}\sum_{k=N_{max}+1}^{N_{max}+n}\frac{1}{k}\right\} + iK'(n)C_n - K''(n)C_n. \quad (10)$$

On the right-hand side of equations (10), the amplitudes of harmonics with the numbers $n > N_{max}$ are approximated by their asymptotic values

$$\tilde{C}_n = iA_s\exp(in\theta_s)/2\pi n, \quad (11)$$

which correspond to the spectrum of a sawtooth wave with the amplitude A_s and the discontinuity at the point θ_s. Such a replacement allows one to analytically calculate the infinite sums on the right-hand side of equations (5). The quantities A_s and θ_s are determined from the values of the two last spectral components of the system, $C_{N_{max}-1}$ and $C_{N_{max}}$, on the assumption that, at $n \approx N_{max}$, the form of the spectrum differs little from its high-frequency asymptotics

$$A_s = 2\pi N_{max}|C_{N_{max}}|, \quad \theta_s = \arg(C_{N_{max}}/C_{N_{max}-1}). \quad (12)$$

The proposed method allows a fairly accurate modeling of the propagation of waves with narrow shock fronts by using a limited number of harmonics $N_{max} = 30$–50.

Equations (10) were numerically integrated by the Runge-Kutta scheme with a fourth-order precision at $N_{max} = 50$. For each subsequent step of calculation in z, the values of $A_s(z)$ and $\theta_s(z)$ were reconstructed by formulas (12) from the values of the coefficients $C_{N_{max}-1}$ and $C_{N_{max}}$ calculated at the preceding step.

The important characteristics that determine the thermal or cavitation effect on the tissue are the temporal wave profile, the behavior of the wave intensity in the medium, and the structure and width of the shock front. The wave profile can be reconstructed as a sum of the smooth and sawtooth components from the numerically calculated amplitudes of the first N_{max} harmonics:

$$V(\theta, z) = \sum_{n=-N_{max}}^{N_{max}} (C_n(z)\exp(-in\theta)) - \frac{A_s}{\pi}\sum_{n=1}^{N_{max}}\frac{\sin(\theta-\theta_s)}{n} + \frac{A_s}{2}\begin{cases} -1 - \dfrac{\theta-\theta_s}{\pi}, & 0 \le \theta \le \theta_s \\ 1 - \dfrac{\theta-\theta_s}{\pi}, & \theta_s \le \theta \le 2\pi. \end{cases} \quad (13)$$

The mean wave intensity was also calculated: $I(z) = \frac{1}{2\pi}\int_0^{2\pi} V^2(z,\theta)d\theta = \sum_{n=-\infty}^{\infty} |C_n(z)|^2$. With allowance for the asymptotic behavior of harmonics at high frequencies (i.e., at $n > N_{\max}$) (11), the wave intensity can be represented as a finite sum

$$I(z) = C_0^2 + \sum_{n=1}^{N_{\max}}\left(2|C_n(z)|^2 - \frac{A_s^2}{2\pi^2 n^2}\right) + \frac{A_s^2}{12}. \quad (14)$$

From the point of view of medical applications, the study of the propagation of intense harmonic waves (ultrasound therapy) and intense single pulses with shock fronts (extracorporeal lithotripsy) are of most interest. Therefore, we consider an initial acoustic signal in the form of a harmonic wave

$$p(\tau, x = 0) = p_0 \sin(\omega_0 \tau) \quad (15)$$

of frequency 1 MHz and amplitude 0.5–7 MPa, which corresponds to the pressure range used in ultrasound therapy, and a shock pulse with an exponential profile behind a shock front

$$p(\tau, x = 0) = \begin{cases} 0, & \tau < \tau_s \\ p_0 \exp\left(-\frac{\tau - \tau_s}{t_0}\right), & \tau > \tau_s. \end{cases} \quad (16)$$

Here, τ_s is the time of the formation of the pulse shock front; the initial amplitude $p_0 = 3$ MPa and the duration $t_0 = 300$ ns of the pulse were set to be close to the parameters of the pulses used in lithotripters (at the focuser output, away from the focus [17]) or to the parameters of the pulses generated by the photoacoustic method in the plane-wave mode [18].

As was mentioned above, the sound velocity dispersion is relatively small. However, it noticeably affects the profile of the acoustic disturbance. Figure 2a presents the numerically calculated evolution of the profile $V = p/p_0$ of an initially harmonic wave (15) in a medium with a linear frequency dependence of attenuation, $\eta = 1$, which corresponds to the parameters of butanediol (Fig. 1). Curve *1* represents the initial wave profile, and curve *2* represents the wave profile at the distance $z = 0.3$ (8 cm) in the case of linear propagation, $N = 0$. Profiles *3* and *4* are calculated for the same distance in the case of a nonlinear propagation $N = 10$ ($p_0 = 4.5$ MPa) in the absence and presence of the sound velocity dispersion, respectively. One can see that the effect of dispersion manifests itself as an asymmetric distortion of the wave profile: the negative semiperiod becomes sharper, while the positive semiperiod is "protracted" so that the position of the peak of the wave increasingly lags behind the wavefront. Such an asymmetry of the profile is characteristic for media with dispersion, e.g., a relaxing medium [1, 18, 19]. In a dispersion medium, the shock front is shifted at the expense of the faster propagation of higher frequencies forming the discontinuity.

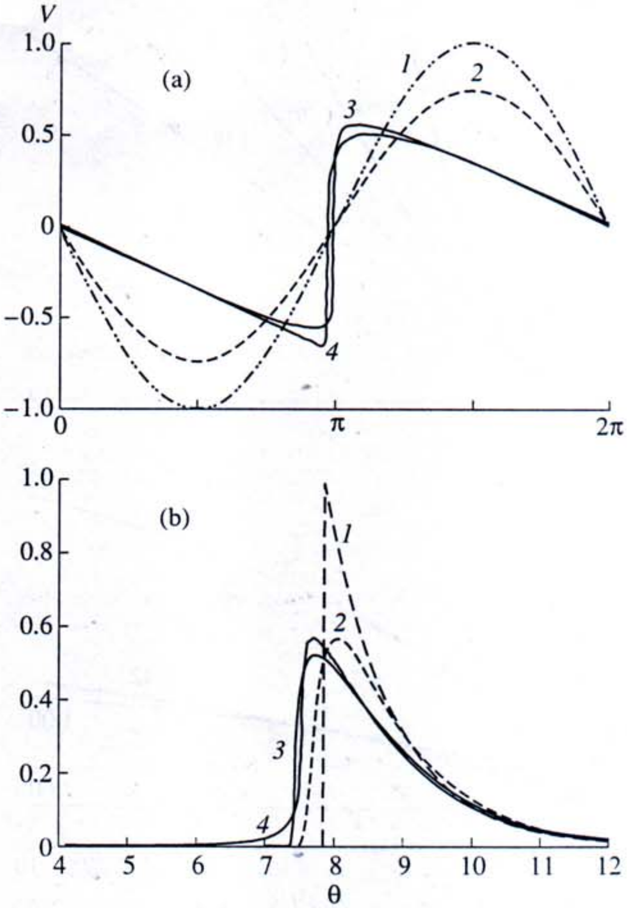

Fig. 2. Effect of nonlinearity, dissipation, and dispersion on the evolution of a wave profile in a medium with $\eta = 1$.
(a) Harmonic initial wave (curve *1*); curves *2–4* correspond to $z = 0.3$: (*2*) profile calculated with allowance for only the attenuation ($N = 0$); (*3*) profile in the presence of attenuation and nonlinearity ($N = 10$); and (*4*) profile in the presence of attenuation, nonlinearity ($N = 10$), and dispersion.
(b) Initial pulse with a shock front (curve *1*); curves *2–4* correspond to $z = 0.23$: (*2*) profile in a linear medium ($N = 0$) with attenuation and dispersion; (*3*) profile in a nonlinear medium ($N = 3.3$) with attenuation and dispersion; and (*4*) profile in a nonlinear medium ($N = 3.3$) with attenuation only.

In a similar way, the dispersion affects the profile of a shock pulse. Figure 2b presents the profile of the initial pulse (16) (curve *1*) and the profiles of the pulse at the distance $z = 0.23$ (6 cm) at which pronounced effects of both nonlinearity and dissipation can be observed. In the case of linear propagation (curve *2*), the shock front broadens because of the attenuation of the high-frequency components. In the case of a nonlinear propagation (curve *3*), the front of the pulse propagates faster, and the front width is less than in the case of linear propagation. Such a manifestation of nonlinear effects is well known. The aforementioned curves *2*

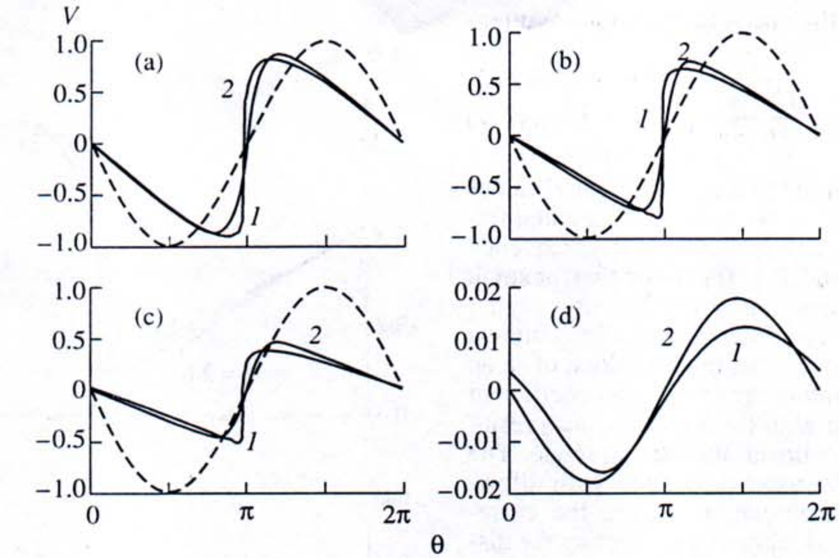

Fig. 3. Profiles of an initially harmonic wave (dashed curves) at different distances z = (a) 0.13, (b) 0.22, (c) 0.4, and (d) 3 in a nonlinear medium with different frequency power laws governing the attenuation: η = 1 and 2 (numbers near the curves). The nonlinear parameter is N = 10.

and *3* are calculated with allowances made for the sound velocity dispersion. To illustrate the role of dispersion, we also present a nonlinear pulse profile calculated with allowance for the presence of attenuation but in the absence of dispersion (curve *4*). Correlating profiles *3* and *4*, one can see that the neglect of dispersion leads to the appearance of a lengthy precursor propagating faster than the shock front, which is a result of the violation of the causality principle. Besides, in the model without dispersion, the peak pressure of the signal is slightly overestimated, and the velocity of the shock front propagation is reduced.

In biological tissues, the frequency dependences of the attenuation coefficient vary from a linear law to an almost square one. The exponent of this power law is one of the parameters that affect the nonlinear evolution of the acoustic signal profile. This effect is illustrated in Fig. 3, which compares the wave profiles formed at different distances in media with linear (η = 1) and a square-law (η = 2) frequency dependences of the attenuation coefficient. The profiles are obtained for an initially harmonic wave. In the medium with the linear law (η = 1), the shock front is formed earlier, it is narrower, and it lasts longer than in the medium with the square law (η = 2). This occurs because in the medium with the square law, the high-frequency harmonics of the spectrum experience a stronger attenuation.

At large distances $x \gg x_{at}$ (Fig. 3d) where the wave reverts to the harmonic form, the peak pressure amplitude in the medium with η = 1 is less than in the medium with η = 2. This result may seem unexpected at first glance, because, in the medium with η = 2, all higher harmonics propagate with higher attenuation coefficients than in the medium with η = 1. The effect is caused by the less intense generation of high-frequency spectrum components in the more dissipative medium, and, hence, by a lower attenuation at the arising shock fronts. Such an effect, which consists of the limitation of the efficiency of the wave energy redistribution toward the high-frequency spectrum region with increasing attenuation of the harmonics of the initial signal, can be used for controlling nonlinear wave interactions. This phenomenon should be most pronounced in a medium with resonance absorption at the second harmonic: in such a medium, the energy transfer to higher frequencies of the spectrum is limited, and the wave propagates almost without distortion [20].

For an initially harmonic wave, the change in the frequency dependence of attenuation from linear to a square-law one leads to some broadening of the shock front, which is illustrated in Fig. 4a. This figure presents the wave profiles calculated for the distance z = 0.5 at N = 15, for media with different exponents η (the numbers near the curves) and the sound velocity dispersion corresponding to these media. One can see that an increase in η has practically no effect on the wave amplitudes, while a noticeable broadening of the shock front is observed. The front width increases several times as the frequency power law governing the attenuation changes from linear to a square one.

The exponent of the frequency power law also noticeably affects the absorption of the total wave energy in a nonlinear medium. Figure 4b shows the variation of the effective normalized attenuation coefficient α_{eff} with distance, where the normalization of the coefficient is performed with respect to its value in the case of a linear wave propagation, and the quantity

α_{eff} is calculated from the variation in the wave intensity (14):

$$\alpha_{eff} = -\frac{dI/dx}{2\alpha_0 I}. \quad (17)$$

Different curves correspond to media with equal values of the attenuation coefficient α_0 at the fundamental frequency, but with different exponents of the frequency power law: $\eta = 1$, 1.5, and 2. One can see that, at small distances, the effective attenuation coefficient is higher for the square law because of the stronger dissipation at high frequencies. However, after the formation of steep segments in the wave profile, the attenuation coefficient is higher in the medium with the linear law as a result of the more efficient nonlinear attenuation mode. The curves shown in Fig. 4b were calculated with allowances made for the dispersion. However, the corresponding dependences calculated by neglecting the dispersion proved to be virtually identical to those shown in Fig. 4b. Hence, although the dispersion affects the wave profile, it has no noticeable effect on the absorption of the wave energy.

The power law (2) is characterized by two parameters, α_0 and η, which play different roles in the evolution of the acoustic signal. Figure 5a shows the profiles of pulse (16) at the distance $x = 6$ cm for media with different exponents of the power law (2) and a fixed parameter α_0 corresponding to the attenuation in butanediol at a frequency of 1 MHz. From this figure, one can see that an increase in the exponent leads to a noticeable broadening of the shock front, which can be explained by a stronger attenuation of the high-frequency spectrum components. At the same time, the variation in the exponent has no noticeable effect on the peak pressure value. By contrast, a change in the parameter α_0 leads to a change in the pulse amplitude. As the attenuation coefficient α_0 increases, the peak pressure decreases, while the width and the position of the shock front vary insignificantly (Fig. 5b). Figure 5b shows the pulse profiles corresponding to the same distance $x = 6$ cm for media with a linear frequency dependence of the attenuation coefficient and different values of α_0 (numbers near the curves).

Thus, nonlinear interactions that occur in both periodic waves and pulsed fields depend on the parameters of the power law governing the frequency dependence of attenuation, especially for strongly nonlinear waves with discontinuities. Even a weak dispersion of sound velocity leads to noticeable distortions of the wave profile as compared to the case of a square-law frequency dependence of attenuation in the absence of dispersion. At the same time, the dispersion has virtually no effect on the dissipation of the energy of a nonlinear wave. The parameters of the power law governing the frequency dependence of attenuation affect the distortion of the profile of a propagating signal in different ways, especially in the case of a shock pulse: the exponent variations mainly affect the width of the pulse shock front, while the variations in the attenuation coefficient mainly affect the peak pressure value. This result allows one to discuss a fundamental possibility of the diagnostics of the parameters of attenuation in the medium by the form of a shock-wave pulse transmitted through it.

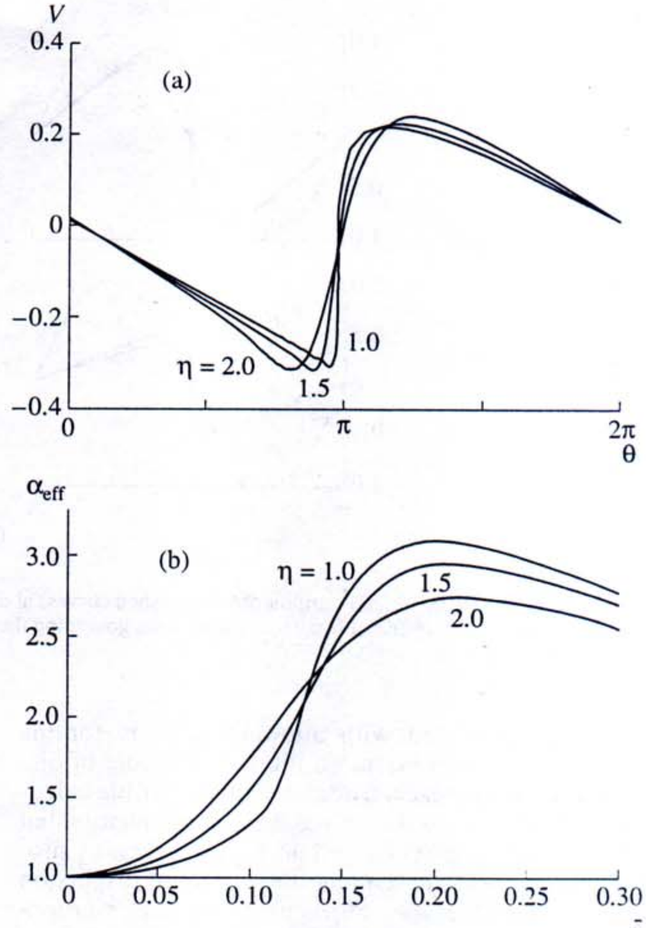

Fig. 4. Effect of the exponent η on the evolution of a wave in a nonlinear medium. At the input, a harmonic wave of frequency 1 MHz is set; $\alpha_0 = 0.038$ cm^{-1}; the nonlinear parameter is $N = 15$. (a) Wave profile at the distance $z = 0.5$; (b) dependence of the effective attenuation coefficient on distance.

In closing, we discuss the problem of stability of a discontinuous wavefront propagating in a lossy medium obeying a frequency power law $\alpha(\omega) \sim \omega^\eta$. It has been found that the shock front of a wave propagating in a medium obeying a square law with $\eta = 2$ is no mathematical discontinuity but has a finite width determined by the viscosity of the medium and the amplitude of the wave [1]. On the other hand, in all media with constant attenuation ($\eta = 0$), the dissipation does not preclude discontinuities in the wave profile [21]. Discontinuities are also stable in relaxing media. Such media are characterized by a constant value of the attenuation coefficient at high frequencies [19]. As far

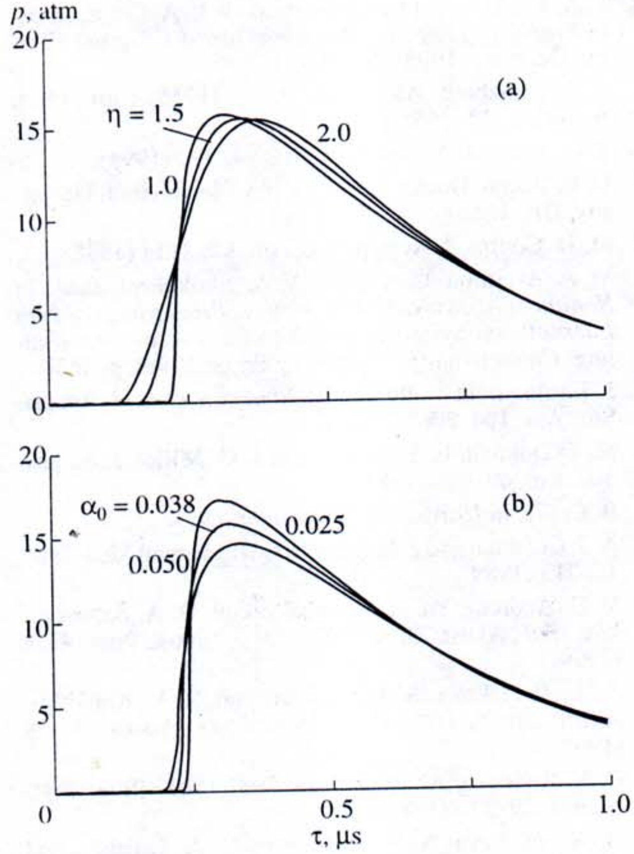

Fig. 5. Effect of the parameters of the power law (2), α_0 and η, on the profile of the shock pulse transmitted through a layer of a nonlinear lossy medium. The input parameters of the pulse are $p_0 = 30$ atm and $t_0 = 0.3$ μs; the distance in the medium is $x = 6$ cm.

(a) Pulse profiles in media with equal attenuation coefficients $\alpha_0 = \alpha(1\text{ MHz}) = 0.038\text{ cm}^{-1}$ and different exponents $\eta = 1, 1.5$, and 2.

(b) Pulse profiles in media with equal exponents $\eta = 1$ and different values of the attenuation coefficient $\alpha_0 = 0.025$, 0.038, and 0.05 cm^{-1}.

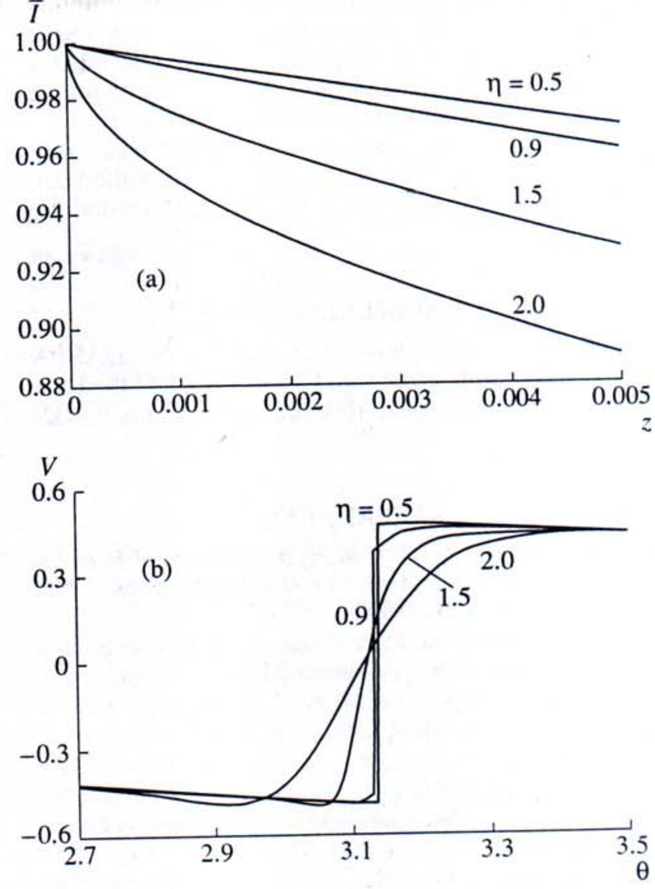

Fig. 6. Stability of the shock front of an initially sawtooth wave in media with different exponents η of the power law (2). (a) Dependence of the normalized intensity $\bar{I} = I(z)/I(z = 0)$ on the distance z; (b) wave profile in the shock region at $z = 0.005$.

as we know, for the case of an arbitrary power law (2), the problem of the stability of a discontinuity had never been studied. Below, we demonstrate that the exponent $\eta = 1$ is a critical value for a discontinuity, i.e., a discontinuity is unstable at $\eta \geq 1$.

Let a plane wave with a discontinuous wave profile be set at the medium input. Without loss of generality, we can assume that the wave is a periodic one. The dissipation of the wave energy is described by the expression

$$\frac{dI}{dx} = \sum_{n=1}^{\infty} 2\alpha_n I_n, \qquad (18)$$

where $I_n \sim |C_n|^2$ and $\alpha_n = \alpha(n\omega_0) \sim n^\eta$ are the intensity of the nth harmonic and the corresponding attenuation coefficient, respectively. At high frequencies, the spectrum of a discontinuous wave always has the asymptotic form (11), i.e., $|C_n| \sim n^{-1}$. Hence, the decrease in the wave energy with distance is determined by a series whose terms at large n behave as $n^{\eta - 2}$. At $\eta \geq 1$, series (18) diverges, i.e., the quantity dI/dx is infinite. If we assume that the high-frequency asymptotics (11) of the spectrum exists in an arbitrarily small interval of distances, we obtain an infinite value of the absorbed energy, which is impossible because of the finiteness of the intensity of the initial wave. Thus, in this case, the discontinuity is unstable. At $\eta < 1$, series (18) converges, and, hence, the existence of the discontinuity is possible.

As an illustration, Fig. 6 presents (a) the numerically calculated dependences of the intensity of an initially discontinuous wave on the distance z and (b) the profile of the wave in the shock region formed at $z = 0.005$ in a nonlinear medium ($N = 10$) for different exponents of the frequency power law governing the attenuation: $\eta = 0.5$, 0.9, 1.5, and 2. The initial wave was taken to be a sawtooth one with the discontinuity amplitude $A_s = 1$, and the intensity was normalized to its initial value at $z = 0$. One can see that, at $\eta < 1$, the derivative dI/dz at $z = 0$ is finite, and the wave profile remains discontinuous in the course of the wave propagation. At $\eta > 1$, the intensity at $z = 0$ decreases infinitely fast, and the initial discontinuity is smoothed out. As the exponent η increases, the effect of smoothing out is enhanced.

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