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# Nonlinear Dynamics of a Vapor–Gas Bubble in a Superheated Region of Finite Size

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**Abstract**—A theoretical study of the growth of a spherical vapor bubble in a spherically symmetric superheated region is described. The modeling of bubble dynamics is based on considering the hydrodynamic and thermal processes inside a bubble and the surrounding liquid.

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## INTRODUCTION

A high-intensity focused ultrasound beam can overheat the medium of its propagation and initiate boiling. Corresponding conditions appear in such regimes of ultrasound therapy as thermal destruction and hystotripsis [1]. Absorption of a focused beam produces a localized heated region of millimeter size that coincides with the focal region of beam. If the medium is overheated, i.e., its temperature exceeds that of boiling and it has a microscopic bubble (cavitation nucleus), this bubble can grow to millimeter size in several milliseconds under vapor pressure, or even faster. Such rapid growth increases the mechanical stresses around the bubble resulting in the emission of audible sound. These sounds can be useful in controlling some types of therapy. In addition, bubbles of millimeter size that form upon boiling are strong acoustic diffusers and thus create regions of increased brightness in an ultrasound image. This is extremely useful for reliable visualization of the region of impact. The aim of this work was a theoretical study of vapor–gas bubble dynamics in a locally overheated liquid under conditions of spherical symmetry.

## BASIC EQUATIONS

As in the approach described in [2], the theoretical part is based on equations of mass, momentum and energy balance, and equation of state, along with the laws of heat and mass transfer [3, 4]. In deriving equations of bubble dynamics, it is assumed that the liquid being pushed by an expanding bubble is incompressible. Although the overheated regions formed by a high-intensity focused ultrasound beam are ellipsoids, it is logical to simplify the problem by considering spherically symmetric overheated regions. In such an approximation, and because the initial nucleus of boiling is in the center of overheated region, the problem becomes spherically symmetric. The equation for liquid motion around a bubble has a Rayleigh–Plesset-type form for the radius of the bubble [5, 6]. The heat transfer model considers both the liquid and the gas phases. The acoustic pressure created by a growing bubble can, like other parameters characterizing its growth, be calculated using a set of equations that joins them with one another.

### *Equations of Basic Parameter Evolution*

Let us consider a set of first order differential equations for four independent variables that depend only on time: bubble radius $R$, velocity of the bubble wall $V$, pressure inside the bubble $p_i$ and number of vapor moles inside the bubble $n_v$ [2]:

$$\dot{R} = V, \tag{1}$$

$$\dot{V} = k_{2V}\left\{\frac{k_{1V}}{C}\left(\dot{p}_i + \frac{2\sigma}{R^2}V + \frac{4\mu}{R^2}V^2\right) + \frac{1}{R(1-V/C)}\left[\left(1+\frac{V}{C}\right)H - \frac{3}{2}\left(1-\frac{V}{3C}\right)V^2\right]\right\}, \tag{2}$$

$$\dot{p}_i = \gamma p_i\left(\frac{\dot{n}_v}{n_v} - \frac{3V}{R}\right) + (\gamma - 1)\frac{3k_g}{R}\frac{T_w - \theta}{\delta_\theta}, \tag{3}$$

$$\dot{n}_v = 4\pi R^2 \frac{\hat{\sigma}}{\sqrt{2\pi M \mathrm{R} T_w}}(p_{\text{sat}} - p_v). \tag{4}$$

The denotations used in equations (1)–(4) are $k_{1V} = \left[p_*/(p_* + p_w - p_0)\right]^{1/\Gamma}/\rho_0$, and $k_{2V} = \left[1 + 4\mu k_{1V}/(CR)\right]^{-1}$. Parameters of the liquid are also introduced: $p_* = \rho_0 c_0^2/\Gamma$ is the inner pressure; $p_w = p_i - 2\sigma/R - 4\mu V/R$ – is the pressure on the bubble wall; $\sigma$ is a surface tension coefficient; $c_0$ is the speed of sound; $C = \sqrt{c_0^2 + (\Gamma - 1)H}$ is the speed of sound on the bubble wall; $\Gamma$ is an empirical constant ($\Gamma = 6.5$ for water); $\rho_0$ is density; $p_0$ is the pressure away from the bubble; and $\mu$ is viscosity. The gas component of the bubble is described by other parameters: $\gamma$ is the adiabatic exponent; $k_g$ is the heat conductivity of the gas; $\theta$ is temperature (which is assumed to be constant inside the bubble, except for a boundary layer near its wall); $\delta_\theta$ is the thickness of the thermal boundary layer; $M$ is the molar weight of the vapor; $R$ is the universal gas constant; $p_v$ is the partial pressure of the vapor; and $p_{sat}$ is the pressure of the saturated vapor. The enthalpy of the liquid on the bubble's wall is expressed as

$$H = \frac{\Gamma}{\Gamma - 1}\frac{p_*}{\rho_0}\left[\left(1 + \frac{p_w - p_0}{p_*}\right)^{(\Gamma-1)/\Gamma} - 1\right]. \qquad (5)$$

### *Equations for the Temperature on the Bubble's Boundary*

To consider a nonuniformly overheated liquid, we must add an equation for the evolution of the liquid's temperature $T_w$ on the bubble boundary to fix (1)–(4). This can be obtained by solving the heat conductivity and energy balance equations on the bubble's boundary. The latter can be written as

$$4\pi R^2\left(k_g\frac{T_w - \theta}{\delta_\theta} - k_L\left.\frac{\partial T}{\partial r}\right|_{r=R}\right) + \dot{n}_v L = 0. \qquad (6)$$

Here, $k_L$ is the heat conductivity of the liquid; $\partial T/\partial r|_{r=R}$ is the spatial derivative of the liquid's temperature on the bubble boundary; and $L$ is the heat of vaporization. As in Eq. (3), the temperature gradient in the boundary layer is approximated by finite difference $(T_w - \theta)/\delta_\theta$.

The solution to the problem of heat conductivity in the liquid can be written as a sum of two solutions of simpler problems: initial and boundary. The temperature will then consist of two corresponding parts: $T = T_1 + T_2$. Let us assume the initial distribution of the temperature in the overheated region is Gaussian: $T|_{t=0} = T_{00}\exp\left(-r^2/r_{00}^2\right)$, where $r$ is a radial coordinate; $r_{00}$ is the radius of the overheated region; and $T_{00}$ is the temperature in the center of region. Using the approximations in [5, 6], we obtain the following expressions for temperature:

$$T_1 = \frac{T_{00}}{\sqrt{4\pi D\tau}}\int_0^\infty dh' e^{-\frac{\left(3h' + R_0^3\right)^{2/3}}{r_{00}^2}}\left[e^{-\frac{(h-h')^2}{4D\tau}} - e^{-\frac{(h+h')^2}{4D\tau}}\right], \qquad (7)$$

$$T_2 = \frac{h}{2\sqrt{\pi}}\int_0^\infty d\tau'\frac{T_w(\tau')}{\left[D(\tau - \tau')\right]^{3/2}}e^{-\frac{h^2}{4D(\tau-\tau')}}. \qquad (8)$$

We introduce new variables here: $h = \left(r^3 - R^3\right)/3$ and $\tau = \int_0^t R^4(t')\,dt'$. In addition, $R_0 = R(0)$ is the initial radius of the bubble; $D = k_L/(\rho_0 c_p)$ is the heat conductivity coefficient; and $c_p$ is the heat capacity at constant pressure. The solutions to Eqs. (7) and (8) allow us to calculate the spatial derivative of the liquid's temperature on the bubble's boundary: $\partial T/\partial r|_{r=R}$. Since this value is also determined according to Eq. (6), the expression for $\dot{T}_w$ can also be calculated numerically as well.

## GROWTH OF A VAPOR BUBBLE IN A UNIFORMLY OVERHEATED LIQUID

To approximate the typical behavior of a vapor–gas bubble, let us consider a simplified problem in which the liquid's temperature is constant in time and space. The liquid is assumed to be nonviscous; i.e., $\mu = 0$. In addition, we ignore changes in pressure inside the bubble ($\dot{p}_i = 0$), i.e., the effect of vapor cooling upon evaporation and the force of surface tension, which is weak in comparison to the internal pressure ($\sigma = 0$). Only two equations are then left: one for the bubble's radius (1) and one for the velocity of the bubble's wall, as follows from (2):

$$\dot{V} = \frac{1}{R(1 - V/C)}\left[\left(1 + \frac{V}{C}\right)H - \frac{3}{2}\left(1 - \frac{V}{3C}\right)V^2\right]. \qquad (9)$$

Figure 1 shows the results from our numeric modeling of Eqs. (1) and (9). The initial radius of the bubble in the three equations is $R_0 = 10$ μm. Other values of $R_0$ were also considered, but the resulting curves of the bubble's radius vs. time during the growth stage, when the bubble's radius was much larger than the initial radius, were virtually independent of any value of $R_0$. The bubble's radius was characterized by almost linear growth and, as was expected, the rate of growth rose with the temperature of the liquid.

This numeric result can be verified analytically by assuming the liquid is incompressible. The equation for enthalpy is then written as $H = (p_w - p_0)/\rho_0$, and the equation of bubble dynamics becomes the familiar Rayleigh equation

$$R\ddot{R} + 3\dot{R}^2/2 = (p_v - p_0)/\rho_0. \qquad (10)$$

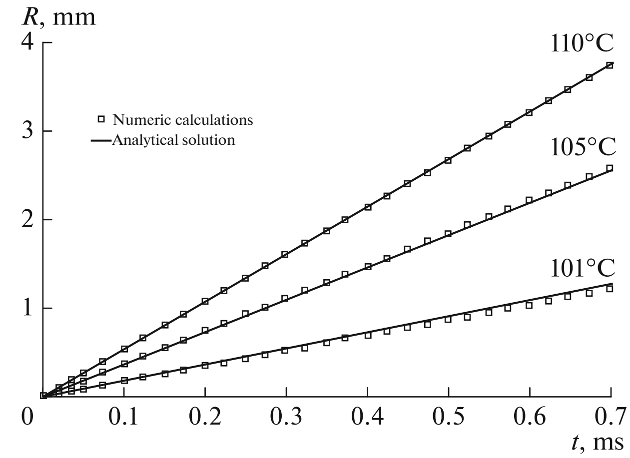


**Fig. 1.** Dependence of a bubble's radius on time at different temperatures of the liquid.

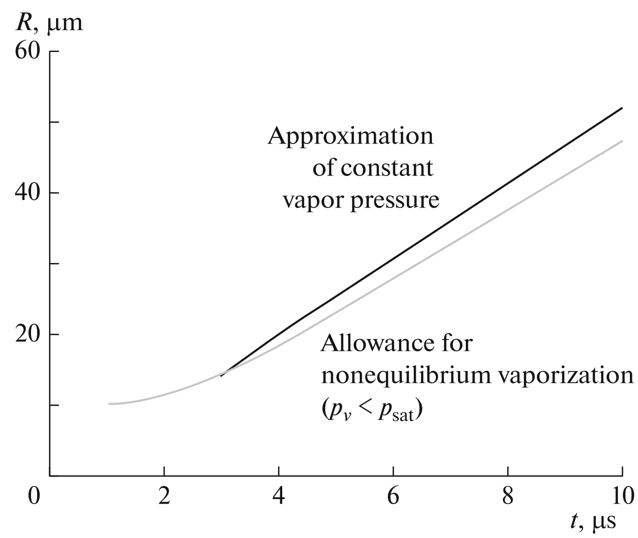


**Fig. 2.** Dependence of bubble's radius on time without and with taking non-equilibrium vaporization into account.

If the heat conductivity in the liquid and gas is strong enough for the bubble's temperature to even out during its growth, and vaporization is fast enough for the vapor inside the bubble to be saturated, the right part of Eq. (10) can be considered independent of time. It can then be solved in a manner similar to the Rayleigh solution for a collapsing empty space [7]. The experimental curves in Fig. 1 are in good agreement with the analytical solution.

When we consider the nonequilibrium vaporization described by Eq. (4), the rate of bubble growth slows (see Fig. 2). This is because vapor forms with a delay; as a result, the pressure inside the bubble falls and the speed at which the bubble grows is reduced.

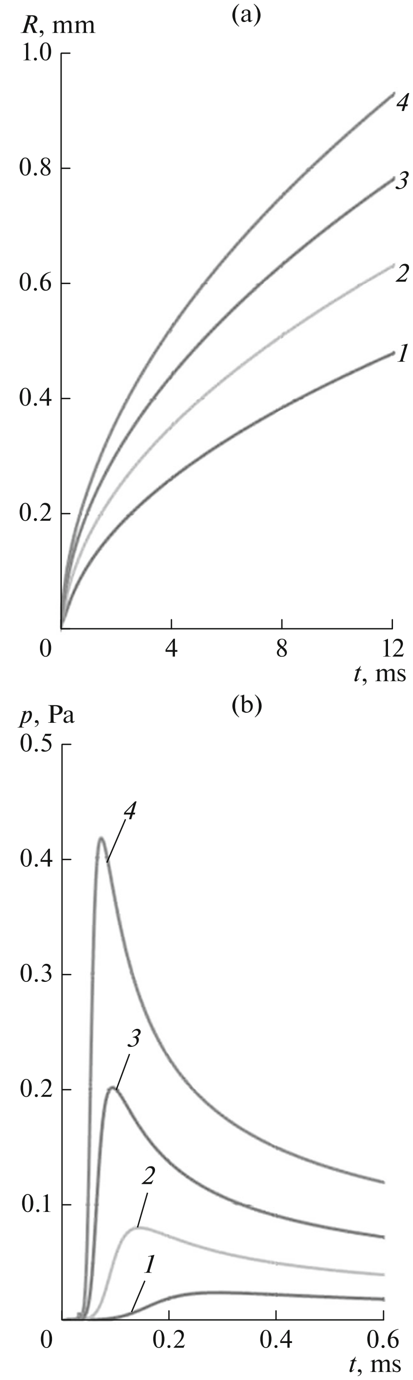


**Fig. 3.** Dependences of (a) a bubble's radius and (b) the acoustic pressure of an emitted wave at distance $r = 5$ cm from the center of the bubble on time at different temperatures of overheating: (*1*) 103, (*2*) 104, (*3*) 105, (*4*) 106°C.

The effect of heat conductivity is apparent in the slowdown of bubble growth due to the marked drop in vapor pressure as a result of its cooling because of its increased volume. We performed numeric calculations with allowance for vaporization and heat conductivity in different cases of liquid overheating (see Fig. 3a) using a program based on the algorithm in [2].

## EMISSION OF SPHERICAL WAVES BY A GROWING VAPOR BUBBLE

Growing vapor bubbles emit spherical waves. It is known that in such cases [8], the velocity potential can be expressed as $\varphi(r,t) = F(t - r/c_0)/r$, while pressure is expressed through the potential as $p = -\rho_0\partial\varphi/\partial t = -\rho_0\dot{F}(t - r/c_0)/r$. Allowing for boundary condition $\partial\phi/\partial r|_{r=R} = \dot{R}$, we obtain $F(t - R/c_0) = -R^2(t)\dot{R}(t)$. It follows that

$$p(r,t) = \frac{\rho_0}{3r}\frac{d^2(R^3)/dt_R^2}{1 + c_0^{-1}dR/dt_R}. \quad (11)$$

Here, bubble radius $R = R(t_R)$is considered at time $t_R$, when an acoustic disturbance that reaches the point of observation at moment $t = t_R + [r - R(t_R)]/c_0$ is emitted. Note that the velocity of the bubble's wall in this process is much slower than that of sound, so the denominator in the second factor of the right part of the equation can be ignored in calculations.

Figure 3b shows the acoustic pressure curves of a growing bubble for different cases of overheating, while the growth of the bubble itself is presented in Fig. 3a. Calculations for different initial conditions showed that the dependence of maximum pressure on the manner of bubble growth excitation is negligible.

## CONCLUSIONS

The set of equations presented in this work formed the basis for modeling the growth of overheated vapor bubbles under conditions of spherical symmetry. The preliminary results preseneted in Fig. 1 show that even with slight overheating to 101°C, a bubble with an initial radius of 10 μm grows to millimeter size in less than 1 ms. If the liquid's temperature is 110°C, the corresponding time is even shorter: about 0.2 ms. Allowing for nonequilibrium vaporization and heat conductivity slows bubble growth, but it still reaches millimeter size in several milliseconds. In hystotripsis experiments, boiling temperature is reached in several milliseconds, and vapor–gas cavities are observed at the same time [1]. These observations are in good agreement with the numeric results from this work.

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