ISSN 1063-7710, Acoustical Physics, 2018, Vol. 64, No. 3, pp. 299–308. © Pleiades Publishing, Ltd., 2018.
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NONLINEAR ACOUSTICS

# Nonlinear Spherical Standing Waves in an Acoustically Excited Liquid Drop

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Received September 14, 2017

**Abstract**—Nonlinear evolution of a standing acoustic wave in a spherical resonator with a perfectly soft surface is analyzed. Quadratic approximation of nonlinear acoustics is used to analyze oscillations in the resonator by the slowly varying amplitude method for the standing wave harmonics and slowly varying profile method for the standing wave profile. It is demonstrated that nonlinear effects may lead to considerable increase in peak pressure at the center of the resonator. The proposed theoretical model is used to analyze the acoustic field in liquid drops of an acoustic fountain. It is shown that, as a result of nonlinear evolution, the peak negative pressure may exceed the mechanical strength of the liquid, which may account for the explosive instability of drops observed in experiments.

*Keywords:* nonlinear spherical waves, standing waves, harmonic generation, resonator, acoustic fountain
**DOI:** 10.1134/S1063771018030144

## 1. INTRODUCTION

Finite-amplitude standing acoustic waves attract the attention of researchers in connection with the possibility to obtain noticeable nonlinear effects in small volumes of liquid or gas. Most of the studies are devoted to plane waves in closed straight tubes with constant cross sections [1–9]. In theoretical descriptions given in the cited publications, acoustic field was represented by two noninteracting Riemann-type plane waves propagating in opposite directions, and the attention of the authors was mainly focused on the regimes with discontinuous (shock) wave profile formation. As a rule, the authors considered wave processes in a gaseous medium, and, therefore, the resonator walls were assumed to be rigid. This caused favorable conditions for manifestations of acoustic nonlinearity, because, at reflection from the resonator mirrors, waves did not change their form and, in the course of their subsequent counterpropagation, underwent further nonlinear distortions. To study the possibility of obtaining intense standing waves with continuous profiles, some of the authors considered standing waves in tubes with varying cross sections [10–12] and in resonators with concentric spherical and cylindrical walls [13–15].

In this paper, we consider a nonlinear acoustic resonator that fundamentally differs from the devices considered earlier. The resonator is represented by a fluid sphere with a perfectly soft boundary. At the beginning, the sphere is assumed to be acoustically excited at its fundamental or any other low-frequency resonance. We analyze the evolution of a spherically symmetric sound field. We assume that no external sources are present; i.e., we study attenuation of free oscillations.

This kind of problem statement attracted our attention in connection with observation of the behavior of liquid drops in an acoustic fountain, which is an acoustic hydrodynamic phenomenon used in, e.g., ultrasonic humidifiers and inhalers. High-speed photography of acoustic fountains shows that, within some time after the source of ultrasound is turned on, a jet emerges from the liquid and breaks down into a chain of drops of the same size [16]. Then, within a certain time interval, the drops (usually beginning with the upper one) loose their stability for the reason yet unknown and explode leading to atomization of the liquid. Recent experimental studies of drop behavior in an acoustic fountain show that, immediately before the stability loss, a dark point appears at the center of a transparent drop [17, 18]. This indicates a possible disruption of the liquid, i.e., appearance of cavitation. Since, according to estimates, the initial sound pressure level in a drop is below the cavitation threshold, one can expect that, in the course of nonlinear oscillations of an excited drop, acoustic energy concentration occurs at its center. This possibility is analyzed below.

## 2. THEORETICAL MODEL FOR DESRIBING THE EVOLUTION OF ACOUSTIC FIELD IN AN EXCITED DROP

### *2.1. Initial Equations*

Unlike the commonly studied case of plane waves, nonlinear acoustic fields formed under spherically symmetric conditions cannot be described by solutions in the form of superposition of counterpropagating waves (Riemann invariants). Therefore, we cannot seek the solution in the form of a nonlinear wave multiply reflected from the resonator boundary with a wave profile slowly varying in the course of propagation. However, it is possible to use another description of acoustic field by considering it as a standing wave with its structure only slightly varying from period to period. To analyze the laws governing the nonlinear wave process, it is necessary to separate fast variations from slow ones in explicit form. For this purpose, we use a basis consisting of weakly interacting standing waves with different frequencies.

We proceed from the general form of wave equation for particle velocity potential $\varphi$ of acoustic field in liquid or gas with allowance for quadratically nonlinear terms. This equation derived in [19] has often been called Kuznetsov's equation [8, 20]:

$$\begin{aligned}\Delta\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} &= -\frac{b}{\rho_0 c_0^2}\frac{\partial}{\partial t}(\Delta\varphi) \\ &+ \frac{1}{c_0^2}\frac{\partial}{\partial t}\left[\frac{\beta-1}{c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + (\nabla\varphi)^2\right].\end{aligned} \tag{1}$$

Here, $c_0$ is the velocity of sound, $\rho_0$ is the equilibrium density of the medium, $b = \varsigma + 4\eta/3$ is the dissipative factor, $\varsigma$ and $\eta$ are the bulk and shear viscosity coefficients, and $\beta$ is the acoustic nonlinearity parameter of the medium. Potential $\varphi$ completely characterizes the acoustic field. In particular, particle velocity $\mathbf{v}$ and sound pressure $p$ are expressed through the potential as follows:

$$\mathbf{v} = \nabla\varphi, \tag{2}$$

$$p = -\rho_0\frac{\partial\varphi}{\partial t} + \frac{\rho_0}{2c_0^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \frac{\rho_0}{2}(\nabla\varphi)^2. \tag{3}$$

To describe the nonlinear field in an acoustically excited liquid drop, we consider a spherically symmetric solution to Eq. (1) in a spherical volume with an acoustically soft boundary. Let $r$ be the distance from the center of the drop and $a$ be its radius in nonexcited state. We denote the potential at the unperturbed drop surface as $\Phi_0(t) = \varphi(a,t)$ and consider the function $\tilde\varphi = \varphi - \Phi_0$. At the surface $r = a$, we have $\tilde\varphi = 0$. In addition, at $r = 0$, condition $\partial\tilde\varphi/\partial r = \partial\varphi/\partial r = 0$ is satisfied, because, by virtue of symmetry, the liquid at the center of the drop is immobile. Under aforementioned boundary conditions, function $\tilde\varphi(r,t)$ within interval $0 \le r \le a$ can be expanded in basis functions $\sin(k_n r)/(k_n r)$, where $k_n = \pi n/a$. Each of the functions describes a standing wave. The aforementioned basis is full and orthogonal if a scalar product of two basis functions is interpreted as the integral of their product over the drop volume. Thus, in the general case, the particle velocity potential can be represented as a superposition of standing waves:

$$\varphi = \Phi_0(t) + \sum_{n=1}^{\infty}\Phi_n(t)\frac{\sin(k_n r)}{k_n r}, \tag{4}$$

where $\Phi_n$ are weighting factors depending on time alone. Note that function $\Phi_0(t)$ is not independent: it can be expressed through the remaining weighting factors $\Phi_n$ because of the condition that sound pressure at the drop boundary is zero (see below). In the course of oscillations, the boundary moves, and, therefore, in solving the nonlinear problem, it is necessary to take into account that sound pressure is zero at $r = a + \xi(t)$, where $\xi$ is the surface displacement. From condition $p(a + \xi, t) = 0$, in quadratic approximation, we obtain $p(a,t) + \xi(\partial p/\partial r)\big|_{r=a} \approx 0$, which, with allowance for the equation of motion, yields boundary condition $(p - \rho_0\xi\partial v/\partial t)\big|_{r=a} \approx 0$, where $v = \partial\varphi/\partial r$ is the radial component of particle velocity. Note that $v(a,t) = d\xi/dt$, and, therefore, from Eqs. (2)–(4), in quadratic approximation, we derive

$$\frac{d\Phi_0}{dt} \approx -\frac{1}{2a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\left(\Phi_m\Phi_n + 2\Pi_m\frac{d\Phi_n}{dt}\right), \tag{5}$$

where $\Pi_m(t)$ is the antiderivative of function $\Phi_m(t)$.

### *2.2. Solution of the Problem by the Slowly Varying Amplitude Method*

We substitute expansion (4) in Eq. (1) and, using Eq. (5), ignore the terms of third and higher orders of smallness:

$$\begin{aligned}&\frac{d^2\Phi_0}{dt^2} + \sum_{n=1}^{\infty}\left(\frac{d^2\Phi_n}{dt^2} + \frac{bk_n^2}{\rho_0}\frac{d\Phi_n}{dt} + k_n^2c_0^2\Phi_n\right)\frac{\sin(k_n r)}{k_n r} \\ &= -\frac{d}{dt}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\left\{\begin{aligned}&\frac{\beta-1}{c_0^2}\frac{d\Phi_m}{dt}\frac{d\Phi_l}{dt}\frac{\sin(k_m r)}{k_m r}\frac{\sin(k_l r)}{k_l r} \\ &+\Phi_m\Phi_l\frac{\partial}{\partial r}\left[\frac{\sin(k_m r)}{k_m r}\right]\frac{\partial}{\partial r}\left[\frac{\sin(k_l r)}{k_l r}\right]\end{aligned}\right\}.\end{aligned} \tag{6}$$

Multiplying both sides of this equality by function $2k_m r\sin(k_m r)/a$, we integrate over the radial coordinate from 0 to $a$. Using orthogonality condition $\frac{2}{\pi}\int_0^{\pi}dx\sin(mx)\sin(nx) = \delta_{mn}$, Eq. (5), and trigonometric identities, we perform some cumbersome but simple transformations to obtain a system of coupled equations in factors $\Phi_n$:

$$\frac{d^2\Phi_n}{dt^2} + 2\delta_n \frac{d\Phi_n}{dt} + \omega_n^2\Phi_n = \frac{d}{dt}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}\Psi_{nml}. \qquad (7)$$

Here, coefficients $\omega_n = k_n c_0$ have the meaning of resonance frequencies and $\delta_n = b\omega_n^2/\left(2\rho_0 c_0^2\right)$ have the meaning of mode damping decrements. In addition, on the right-hand side, the following notation is used:

$$\Psi_{nml} = -(-1)^{n+m+l}\frac{1}{a^2}\left(\Phi_m\Phi_l + 2\Pi_m\frac{d\Phi_l}{dt}\right) + \frac{n}{2\pi ml}\left(S_{n+m+l} - S_{n+m-l} - S_{n-m+l} + S_{n-m-l}\right) \times\left[\frac{\beta-1}{c_0^2}\frac{d\Phi_m}{dt}\frac{d\Phi_l}{dt} - \left(\frac{\pi}{a}\right)^2\Phi_m\Phi_l\frac{n^2-m^2-l^2}{2}\right], \qquad (8)$$

where coefficients $S_n$ are discrete values of integral sine:

$$S_n = \mathrm{Si}(\pi n) = \int_0^{\pi} dx\frac{\sin(nx)}{x} = \int_0^{\pi n} dx\frac{\sin x}{x}. \qquad (9)$$

One can see that each of the equations of system (7) has the form of a classical oscillator equation with its right-hand side describing a quadratically nonlinear, i.e., weak, source.

The solution can be analyzed using the slowly varying amplitude method. In an ideal linear medium, solutions to Eqs. (7) have the form of constant-amplitude harmonic oscillations:

$$\Phi_n = \frac{C_n}{2}e^{-i\omega_n t} + \frac{C_n^*}{2}e^{i\omega_n t}. \qquad (10)$$

Effect of dissipation and nonlinearity leads to slow variation of amplitudes $C_n$ with time. In this case, accurate to the terms of second order of smallness, the derivative of the potential at the drop boundary can be represented in the form

$$\frac{d\Phi_0}{dt} \approx \frac{1}{4a^2}\left[\sum_{n=1}^{\infty}C_nC_n^* + \sum_{n=1}^{\infty}\left(\frac{D_n}{2}e^{-i\omega_n t} + \frac{D_n^*}{2}e^{i\omega_n t}\right)\right],$$

where harmonic component amplitudes $D_n$ are expressed through $C_n$ as follows:

$$D_n = (-1)^n\left\{2\sum_{m=1}^{\infty}\frac{n^2+m^2+mn}{m(n+m)}C_m^*C_{m+n} - \sum_{m=1}^{n-1}\frac{n+m}{n-m}C_{n-m}C_m\right\}.$$

Ignoring small terms related to time derivatives of slowly varying quantities, from Eqs. (7) we obtain a system of reduced equations in complex amplitudes $C_n$:

$$\frac{dC_n}{dt} + \delta_nC_n = -\frac{1}{4a^2}\left\{2\sum_{m=1}^{\infty}\left[-\frac{n^2+m^2+mn}{m(n+m)} + \frac{n\beta\pi}{2}\times\left(S_{2n} + S_{2m} - S_{2m+2n}\right)\right]C_m^*C_{m+n} + \sum_{m=1}^{n-1}\left[\frac{n+m}{n-m} + \frac{n\beta\pi}{2}\left(S_{2n} - S_{2m} - S_{2n-2m}\right)\right]C_{n-m}C_m\right\}. \qquad (11)$$

It should be noted that nonlinear effects are not completely determined by acoustic nonlinearity parameter $\beta$ of the medium: the first terms involved in the two bracketed expressions do not depend on this parameter. These terms allow for the fact that, in the nonlinear case, the potential at the drop boundary has a small nonzero value.

However, in cases of practical interest, the second terms appearing in the bracketed expressions on the right-hand side of Eq. (11) far exceed the corresponding first terms. For example, for water ($\beta \approx 3.5$), the second terms exceed the first terms several times even for $n = 1$, and, for large values of $n$, the difference is still greater. Therefore, by ignoring the aforementioned small terms, we obtain a fairly adequate approximation:

$$\frac{dC_n}{dt} + \delta_nC_n = -\frac{n\pi\beta}{8a^2}\left\{2\sum_{m=1}^{\infty}C_m^*C_{n+m}\times\left[S_{2n} + S_{2m} - S_{2(n+m)}\right] + \sum_{m=1}^{n-1}C_mC_{n-m}\left[S_{2n} - S_{2m} - S_{2(n-m)}\right]\right\}. \qquad (12)$$

This approximation corresponds to ascribing zero value to quantity $\Phi_0$ in expansion (4). In this case, according to Eqs. (4) and (10), we have

$$\varphi(r,t) = \sum_{n=1}^{\infty}\left(\frac{C_n}{2}e^{-i\omega_n t} + \frac{C_n^*}{2}e^{i\omega_n t}\right)\frac{\sin(k_nr)}{k_nr}. \qquad (13)$$

For our description, an important criterion of its correctness should be verification of the energy conservation law. The total sound energy in the drop can be expressed in the form

$$E = \pi\rho_0 a\sum_{n=1}^{\infty}|C_n|^2. \qquad (14)$$

This yields $dE/dt = \pi\rho_0 a\sum_{n=1}^{\infty}\left(C_n^*dC_n/dt + C_n\,dC_n^*/dt\right)$. Using expressions for $dC_n/dt$ from Eqs. (11) or (12) and regrouping the corresponding double sums, we obtain $dE/dt = -\pi\rho_0 a\sum_{n=1}^{\infty}2\delta_n|C_n|^2$. Hence, in the presence of viscosity, energy decreases, whereas, for $\delta_n \to 0$, it remains constant (see also Fig. 5). Thus, in the absence of viscosity, both system of equations (11) and its approximate version (12) satisfy the energy conservation law.

### *2.3. Solution of the Problem by the Slowly Varying Profile Method*

Spherical standing waves in a drop can be represented as a superposition of two traveling waves: converging and diverging ones or, still simpler, a single wave periodically reflected from the drop surface and alternately diverging and converging. In this representation, variation of acoustic field in the drop under the effect of dissipation and nonlinearity can be analyzed in the form of a slow evolution of the aforementioned converging–diverging wave profile. In our consideration, slow evolution is interpreted not as waveform behavior in the course of wave propagation in space (spatial inhomogeneity of acoustic field in the drop is significant) but as small variation of the periodically repeated temporal sound pressure profile at every spatial point with time sequentially passing from period to period.

We represent the desired solution for the potential in the form

$$\varphi(r,t) = \frac{F(r,t)}{r}. \qquad (15)$$

From Eq. (1), we obtain a classical one-dimensional equation for $F(r,t)$:

$$\frac{\partial^2 F}{\partial r^2} - \frac{1}{c_0^2}\frac{\partial^2 F}{\partial t^2} = q. \qquad (16)$$

Here, the right-hand side describes the sources determined by nonlinear dissipative processes:

$$q = \frac{\partial}{\partial t}\left\{-\frac{b}{\rho_0 c_0^2}\frac{\partial^2 F}{\partial r^2} + \frac{1}{rc_0^2} \times\left[\frac{\beta-1}{c_0^2}\left(\frac{\partial F}{\partial t}\right)^2 + \left(\frac{\partial F}{\partial r} - \frac{F}{r}\right)^2\right]\right\}. \qquad (17)$$

Function $F(r,t)$ is considered within the interval $0 \le r \le a$. From the finiteness of potential $\varphi$ at the center of the drop and from the above-stated approximate condition of zero potential value at the drop surface, we obtain zero boundary conditions at both ends of the interval:

$$F(0,t) = F(a,t) = 0. \qquad (18)$$

Solution to the wave equation with zero right-hand side has the form of a wave multiply reflected from the boundaries of the interval: $F(r,t) = \psi(t - r/c_0) - \psi(t + r/c_0)$, where function $\psi(t)$ describes the wave profile. With the appearance of a small right-hand side, the aforementioned structure of the solution is retained but profile $\psi(t)$ should slowly vary in the course of multiple reflection because of the effect of weak sources.

The above form of $F(r,t)$ yields the following initial conditions ($t = 0$) for the wave function and its time derivative:

$$F(r,0) = \psi(-r/c_0) - \psi(r/c_0), \qquad (19)$$

$$\frac{\partial F}{\partial t}(r,0) = \dot\psi(-r/c_0) - \dot\psi(r/c_0), \qquad (20)$$

where the dot denotes derivative with respect to argument. Solution to one-dimensional wave equation (16) is expressed by the known D'Alembert formula [21]. For the case of an infinite straight line, it has the form

$$F(r,t) = \frac{F(r+c_0t,0) + F(r-c_0t,0)}{2} + \frac{1}{2c_0}\int_{r-c_0t}^{r+c_0t}\frac{\partial F}{\partial t}(r',0)dr' - \frac{1}{2c_0^3}\int_0^t dt'\int_{r-c_0(t-t')}^{r+c_0(t-t')} q(r',t')\,dr'. \qquad (21)$$

Taking into account initial conditions (19) and (20), we obtain

$$F(r,t) = \psi\left(t - \frac{r}{c_0}\right) - \psi\left(t + \frac{r}{c_0}\right) + F_q(r,t), \qquad (22)$$

where additional perturbation $F_q$ generated by the sources is explicitly expressed through source density $q$:

$$F_q(r,t) = -\frac{c_0}{2}\int_0^t dt'\int_{r-c_0(t-t')}^{r+c_0(t-t')} q(r',t')\,dr'. \qquad (23)$$

The first two terms appearing in solution (22) describe the initial wave multiply reflected from the ends of the segment, and the last term corresponds to the wave excited by source $q$. Here, $q(r,t)$ has support $[0,a]$; i.e., outside this interval, $q = 0$. Now, solving the problem within the segment with boundary conditions (18), we have $F_q(0,t) = F_q(a,t) = 0$ and, following the reflection method, instead of $q(r,t)$, we preset sources $\hat q(r,t)$ in the form of a periodic continuation with period $2a$:

$$\hat q(r,t) = \begin{cases} \ldots \\ q(r+2a,t), & -2a \le r \le -a \\ -q(-r,t), & -a \le r \le 0 \\ q(r,t), & 0 \le r \le a \\ -q(-r+2a,t), & a \le r \le 2a \\ \ldots \end{cases}. \qquad (24)$$

Let us consider the corresponding solution at the center of the drop, i.e., $\varphi_q(t) = \lim_{r\to 0}(F_q(r,t)/r)$. From Eqs. (23) and (24), we obtain

$$\varphi_q(t) = \frac{1}{2}\int_{-c_0t}^{0} dr'\hat{q}\left(r', t + \frac{r'}{c_0}\right) - \frac{1}{2}\int_{0}^{c_0t} dr'\hat{q}\left(r', t - \frac{r'}{c_0}\right). \quad (25)$$

Variation within one period with allowance for periodicity of $\hat{q}$ is as follows:

$$\begin{aligned}\delta\varphi_q(t) &= \varphi_q\left(t + \frac{2a}{c_0}\right) - \varphi_q(t) \\ &= -\int_0^a dr'\left[q\left(r', t - \frac{r'}{c_0}\right) - q\left(r', t + \frac{r'}{c_0}\right)\right].\end{aligned} \quad (26)$$

Let $\varphi_0(\tau,t)$ be the potential at the center of the drop, where newly introduced argument $\tau$ follows the slow variation of the wave profile. As quantity $\tau$, we choose time varying in a discrete manner at a step identical to the period: $\delta\tau = 2a/c_0$. Since the profile variation within one period is small, we change to continuous "slow" time $\tau$ and approximate the derivative by a finite difference:

$$\begin{aligned}\frac{\partial\varphi_0(\tau,t)}{\partial\tau} &\approx \frac{\delta\varphi_q(t)}{\delta\tau} \\ &= -\frac{c_0}{2a}\int_0^a dr'\left[q\left(r', t - \frac{r'}{c_0}\right) - q\left(r', t + \frac{r'}{c_0}\right)\right].\end{aligned} \quad (27)$$

Substituting the expression for $q$ (see Eq. (17)) on the right-hand side of Eq. (27) and taking into account expression $F(\tau,r,t) = \psi(\tau,t - r/c_0) - \psi(\tau,t + r/c_0)$, after some transformation, we arrive at the following nonlinear evolution equation for describing the potential at the center of the drop:

$$\begin{aligned}&\frac{\partial\varphi_0(\tau,t)}{\partial\tau} - \frac{b}{2\rho_0c_0^2}\frac{\partial^2\varphi_0(\tau,t)}{\partial t^2} \\ &= -\frac{\beta}{8ac_0}\frac{\partial}{\partial t}\int_0^a \frac{dr'}{r'}\left[\varphi_0\left(\tau, t - \frac{2r'}{c_0}\right) - \varphi_0\left(\tau, t + \frac{2r'}{c_0}\right)\right] \\ &\times\left[\varphi_0\left(\tau, t - \frac{2r'}{c_0}\right) + \varphi_0\left(\tau, t + \frac{2r'}{c_0}\right) - 2\varphi_0(\tau,t)\right].\end{aligned} \quad (28)$$

Note that potential $\varphi(\tau,r,t) = = [\psi(\tau,t - r/c_0) - \psi(\tau,t + r/c_0)]/r$ outside the center is expressed through the potential at the center, because $\varphi_0(\tau,t) = -(2/c_0)\partial\psi(\tau,t)/\partial t$. Thus, functional equation (28) describes the nonlinear dissipative evolution of acoustic field in the drop, i.e., completely solves the problem under study.

According to Eq. (13), we have

$$\varphi_0(\tau,t) = \sum_{n=1}^{\infty}\frac{C_n(\tau)e^{-i\omega_nt} + C_n^*(\tau)e^{i\omega_nt}}{2}. \quad (29)$$

Unlike the previous section, here, the argument of $C_n$ is denoted by $\tau$ (instead of $t$) to distinguish slow variation from fast one. Substituting expansion (29) in Eq. (28), for slowly varying mode amplitudes $C_n(\tau)$, we obtain Eqs. (12). Thus, descriptions by the two aforementioned methods, namely, the slowly varying amplitude method and the slowly varying profile method, are in full agreement with each other.

### *2.4. Relation between the Equation for Acoustic Field Dynamics in a Drop and the Burgers Equation*

To analyze the system of reduced equations (11) and functional equation (28) related to it, it is convenient to use dimensionless variables. Let $P_0$ be the sound pressure amplitude in the drop initially excited at its lowest resonance frequency $\omega_0 = c_0\pi/a$, so that acoustic field in the drop is $p(r,t) = P_0\sin\omega_0t\,\mathrm{sinc}(\omega_0r/c_0)$. We introduce characteristic times of manifestation for nonlinear and dissipative effects:

$$\tau_{nl} = 4\rho_0c_0^2/(\beta\omega_0P_0), \quad (30)$$

$$\tau_{\mathrm{diss}} = 2\rho_0c_0^2/(b\omega_0^2), \quad (31)$$

and a dimensionless parameter characterizing the competition of these phenomena:

$$\Gamma = \tau_{nl}/\tau_{\mathrm{diss}}. \quad (32)$$

For brevity, we also introduce the notations

$$g_{nm} = \frac{2}{\pi}(S_{2n} + S_{2m} - S_{2m+2n}), \quad (33)$$

$$a_{nm} = \frac{4}{\pi^2\beta}\frac{n^2 + m^2 + mn}{mn(n+m)}, \quad (34)$$

$$b_{nm} = \frac{4}{\pi^2\beta}\frac{n+m}{(n-m)n}. \quad (35)$$

We normalize the slow time by the nonlinear time scale: $z = \tau/\tau_{nl}$. Using the characteristic amplitude value for the potential, $C_0 = P_0/(\omega_0\rho_0)$, we introduce dimensionless harmonic amplitudes $\bar{C}_n = C_n/C_0$. Then, the system of reduced equations (11) takes the form

$$\begin{aligned}\frac{d\bar{C}_n}{dz} + \Gamma n^2\bar{C}_n = -\frac{n}{4}\Bigg\{&2\sum_{m=1}^{\infty}(g_{nm} - a_{nm})\bar{C}_m^*\bar{C}_{m+n} \\ &- \sum_{m=1}^{n-1}(g_{n-m,m} - b_{nm})\bar{C}_{n-m}\bar{C}_m\Bigg\}.\end{aligned} \quad (36)$$

It should be solved under the following initial conditions:

$$\bar{C}_1(0) = 1, \quad \bar{C}_{n>1}(0) = 0.$$

Note that, in coefficients $S_n = \mathrm{Si}(\pi n)$, the argument of integral sine (9) is always much greater than unity. Therefore, we can use approximation $S_n \approx \pi/2$,

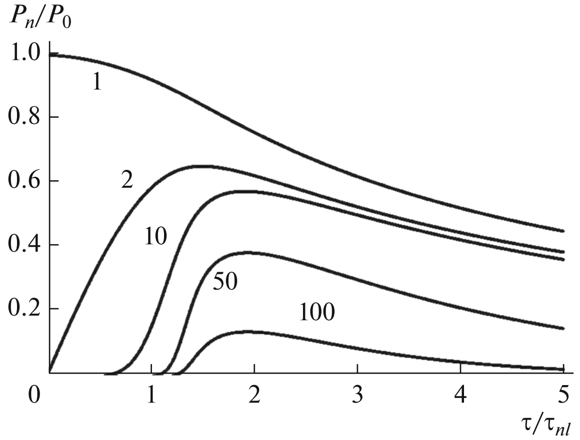


**Fig. 1.** Time dependences of sound pressure harmonic amplitudes at the center of a spherical resonator for $\Gamma = 0.01$. Amplitudes are normalized by initial pressure amplitude $P_0$, and time by characteristic nonlinear scale $\tau_{nl}$. Numbers of harmonics are indicated near the corresponding curves.

i.e., $g_{nm} \approx 1$. Ignoring small coefficients $a_{nm}$ and $b_{nm}$, which corresponds to passage from Eqs. (11) to Eqs. (12), from Eqs. (36) we obtain

$$\frac{d\bar{C}_n}{dz} + \Gamma n^2 \bar{C}_n = -\frac{n}{4}\left\{2\sum_{m=1}^{\infty} \bar{C}_m^* \bar{C}_{m+n} - \sum_{m=1}^{n-1} \bar{C}_{n-m}\bar{C}_m\right\}. \quad (37)$$

Introducing auxiliary quantities $B_n = i\bar{C}_n$, we represent Eq. (37) in the form

$$\frac{dB_n}{dz} + \Gamma n^2 B_n = -\frac{in}{4}\left\{2\sum_{m=1}^{\infty} B_m^* B_{m+n} + \sum_{m=1}^{n-1} B_{n-m}B_m\right\}. \quad (38)$$

One can see that the system of equations (38) is nothing but a system of equations for harmonics of a nonlinear plane wave described by the Burgers equation [22]:

$$\frac{\partial V}{\partial z} - V\frac{\partial V}{\partial \theta} = \Gamma \frac{\partial^2 V}{\partial \theta^2}, \quad (39)$$

$$V(z,\theta) = \sum_{n=1}^{\infty}\left(\frac{B_n}{2}e^{-in\theta} + \frac{B_n^*}{2}e^{in\theta}\right). \quad (40)$$

Here, $\theta = \omega_0 t$ is the dimensionless "fast" time and function $V(z,\theta)$ describes the wave profile. Note that expression $B_n = i\bar{C}_n$ means that functions $V(z,\theta)$ and $U(z,\theta) = \varphi_0(\tau,t)/C_0$ are related to each other through the Hilbert transform:

$$U(z,\theta) = -\frac{1}{\pi}\,\text{v.p.}\int_{-\infty}^{\infty}\frac{V(z,\theta')}{\theta - \theta'}d\theta'. \quad (41)$$

Owing to this relation, many known periodic solutions to Burgers equation (39) can be used to construct approximate solutions in describing the acoustic field in a drop.

## 3. NUMERICAL SIMULATION OF THE EVOLUTION OF ACOUSTIC FIELD IN A DROP INITIALLY EXCITED AT THE FUNDAMENTAL RESONANCE

### *3.1. Spectrum and Time Profile of Sound Pressure at the Center of the Drop*

Spectral amplitudes of the potential at the center of the drop are described by an infinite system of coupled equations (11). In the case of numerical integration, the number of retained harmonics should be restricted. To achieve a negligibly small effect of spectrum truncation on the calculation accuracy, it is necessary to choose a sufficiently large number of harmonics $N$, so that higher harmonic amplitudes be strongly suppressed by dissipative processes. A reasonable estimate follows from the solution to Burgers equation (39) in the form of a Fay series [22]: $N = 2/\Gamma$. In practice, to refine the aforementioned estimate, the number of harmonics was increased until the results of calculations ceased depending on it. Calculations were performed by the finite-difference method in Fortran language using Runge–Kutta method of order 4.

The quantity of practical interest is not potential $\varphi$ but the sound pressure at the center of the drop. The corresponding relation in linear approximation has the form $p = -\rho_0\,\partial\varphi/\partial t$, which yields the relation between pressure harmonic amplitudes and potential: $P_n = i\omega_n C_n$. It is convenient to normalize the pressure harmonic amplitudes by the initial first harmonic amplitude $P_0$, i.e., to introduce $\bar{P}_n = P_n/P_0 = in\bar{C}_n$.

Figure 1 shows the results of calculating some of harmonic amplitudes $|\bar{P}_n|$ as functions of dimensionless slow time $z$ for $\Gamma = 0.01$. Here and below, we assume that $\beta = 3.52$ (water). One can see that active growth of higher harmonics begins at $z \approx 1.2$. With further increase in time $z$, as a result of interaction, different harmonic amplitudes become close in magnitude and then decrease because of viscous absorption.

Harmonic amplitudes allow us to calculate the sound pressure profile at the center of the drop. Figure 2 shows normalized pressure profiles within one oscillation period for $\Gamma = 0.01$. The initial sine-shaped profile is considerably distorted, so that both negative and positive peak pressure values (they are equal in magnitude) increase while the waveform taken within one period acquires the form of a short bipolar pulse. Calculations show that the peak pressure as a function of slow time exhibits a characteristic behavior: first, it increases under the effect of nonlinearity, then reaches

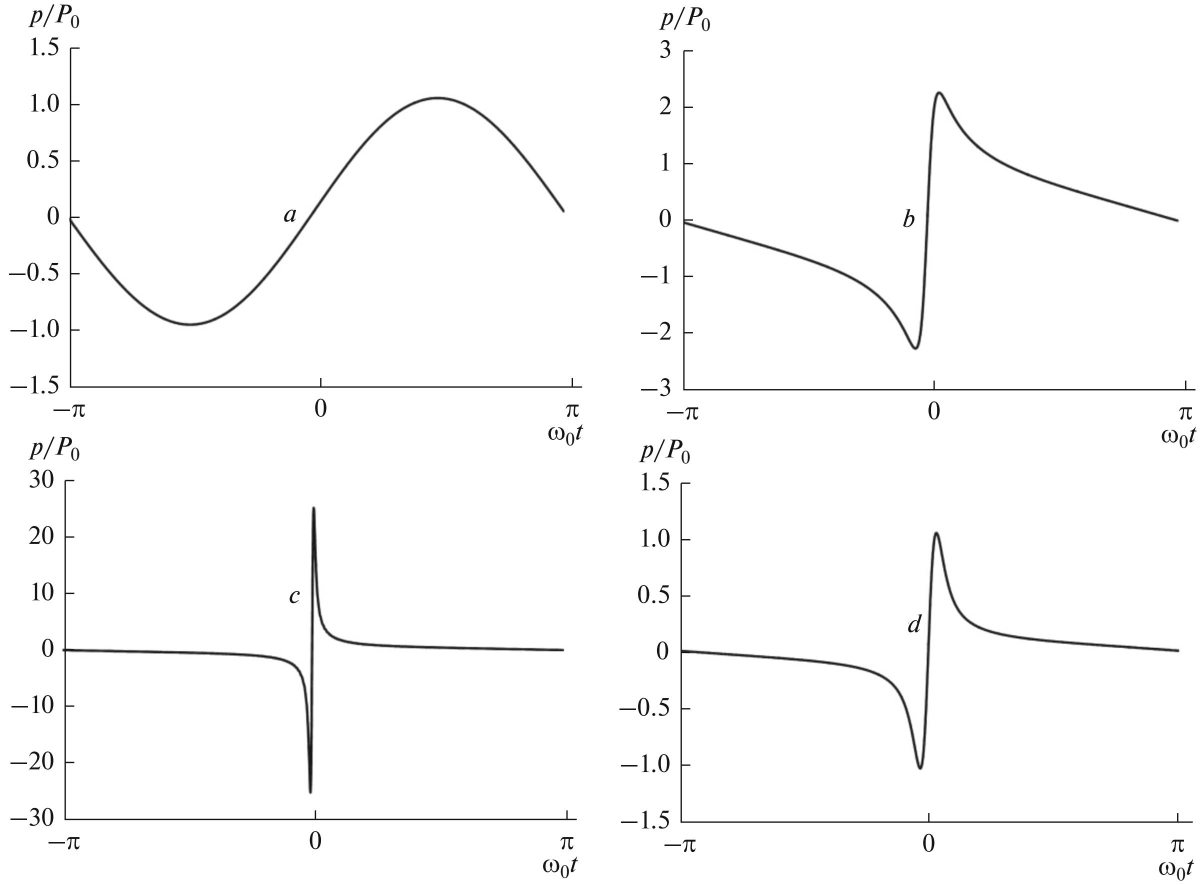


**Fig. 2.** Temporal profile of a single period of sound pressure at the center of a spherical resonator for $\Gamma = 0.01$ and different instants of slow time $\tau$: $\tau/\tau_{nl}$ = (*a*) 0, (*b*) 1, (*c*) 2, and (*d*) 20. Pressure $p$ is normalized by initial amplitude $P_0$.

its maximum $p_{\max}$, and then decreases because of viscous absorption.

The value of maximum peak pressure $p_{\max}$ reached at the center of the drop depends on parameter $\Gamma$. Figure 3 shows the results of calculating quantity $p_{\max}\Gamma/P_0$ as a function of $\Gamma^{-1} \sim P_0$. One can see that, after a small increase in region $1/\Gamma \gg 1$, the curve reaches saturation, which allows us to conclude that the amplification coefficient is adequately approximated by the following simple dependence:

$$\frac{p_{\max}}{P_0} \approx \frac{1}{4\Gamma}. \quad (42)$$

This suggests that the maximum peak pressure reached at the center of the drop in the course of its oscillations depends on the initial amplitude according to square law $p_{\max} \approx P_0^2 \beta/(8b\omega_0)$.

### 3.2. Characteristics of Acoustic Field in the Drop

Above, in describing the wave process in the drop by the slowly varying profile method, we showed that, by introducing auxiliary function $\psi(\tau, t)$ on the basis of equality $\varphi_0(\tau, t) = -(2/c_0)\partial\psi(\tau, t)/\partial t$, it is possible to represent the spatiotemporal profile of particle

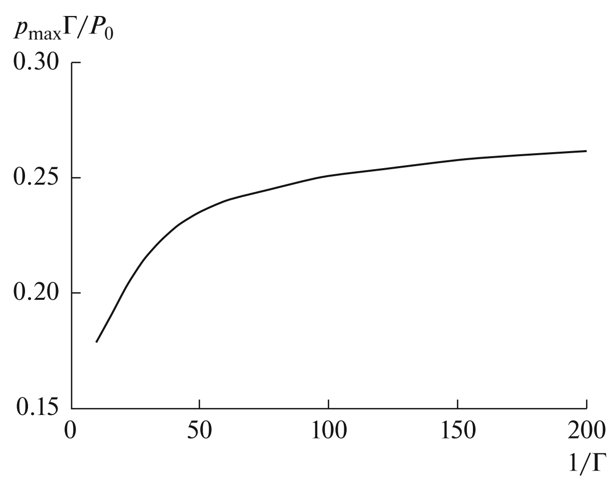


**Fig. 3.** Dependence of normalized maximum peak pressure $p_{\max}\Gamma/P_0$ on $1/\Gamma$.

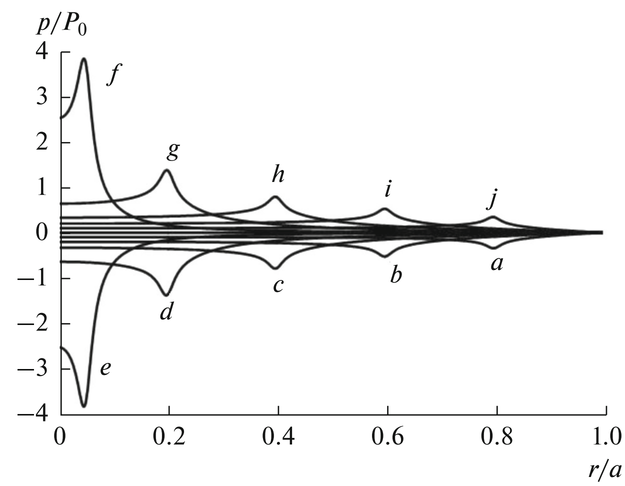


**Fig. 4.** Dependence of normalized sound pressure in the drop on radial coordinate $r$ at sequential instants of fast time $t$ for $\Gamma = 0.01$ and normalized slow time $\tau/\tau_{nl} = 5$. Different wave profiles correspond to normalized instants of time $\omega_0 t/\pi$: (*a*) −0.8, (*b*) −0.6, (*c*) −0.4, (*d*) −0.2, (*e*) −0.05, (*f*) 0.05, (*g*) 0.2, (*h*) 0.4 , (*i*) 0.6 , and (*j*) 0.8. The pressure in the drop becomes zero twice during the period: at $t = 0$ and at $\omega_0 t/\pi = 1$.

velocity potential in the drop by the formula $\varphi(\tau, r, t) = [\psi(\tau, t - r/c_0) - \psi(\tau, t + r/c_0)]/r$. The quantity of practical interest is the sound pressure profile. Using the linear relation between pressure and potential $p = -\rho_0\, \partial\varphi/\partial t$, we obtain

$$p(\tau, r, t) = \frac{\rho_0 c_0}{2} \frac{\varphi_0(\tau, t - r/c_0) - \varphi_0(\tau, t + r/c_0)}{r}. \quad (43)$$

Thus, the problem of determining the acoustic field within the entire drop volume is reduced to the above-described problem of calculating the potential at the center of the drop. Figure 4 shows the spatial profiles of sound pressure for different instants of fast time at fixed slow time $z = 5$ for $\Gamma = 0.01$.

Unlike the case of linear excitation of the drop at fundamental frequency, where the pressure distribution in the standing wave within the interval $0 \le r/a \le 1$ retains its cosine form, in the nonlinear case, we obtain a clearly defined pulse with a sharp peak, which alternately propagates to the right and to the left being reflected from the boundaries of the interval. At reflection from each of the ends, the wave form is inverted, and, as the wave approaches the center of the drop $r = 0$, a considerable increase in peak pressure takes place because of the focusing of the pulsed spherical wave. Note that, unlike the behavior of the finite-amplitude plane wave profile, where the initial sine-shaped profile acquires a saw-tooth shape with shock fronts in the course of propagation, the nonlinear wave distortion in the drop manifests itself in the broken profile at the sharp peak of the pulse.

Earlier, it was noted that the total acoustic energy in the drop is expressed by Eq. (14); i.e., after normalization by the initial energy value $E_0$, we obtain $E/E_0 = \sum_{n=1}^{\infty} |\bar{C}_n|^2$. Figure 5 shows the dependences of this quantity on dimensionless slow time $z$ for several values of $\Gamma$. One can see that, at the initial stage, the energy varies only weakly despite harmonic generation. A decrease begins only for $z > 1$. This kind of energy behavior resembles the corresponding dependence for nonlinear plane waves described by the Burgers equation.

### *3.3. Example of Sound Pressure Calculation for a Drop of an Acoustic Fountain*

Now, we consider a practical example related to previous experimental observation of drops in an acoustic fountain [18]. We assume that the drop diameter is $2a = 1.5$ mm and the parameters of the liquid are as follows: sound velocity $c_0 = 1500$ m/s, density $\rho_0 = 1000$ kg/m$^3$, acoustic nonlinearity parameter $\beta = 3.52$, and effective viscosity coefficient $b = 3.9 \times 10^{-3}$ Pa s (water). We consider regimes with the initial standing wave amplitude at the center of the drop, $P_0$ , being on the order of one megapascal. In this case, characteristic scales (30) and (31) are $\tau_{nl} = 0.4$ ms (for $P_0 = 1$ MPa) and $\tau_{diss} = 29$ ms, which corresponds to $\Gamma \approx 0.014$. Hence, nonlinear distortions are noticeable ($\Gamma \ll 1$) and develop within one millisecond or less.

Figure 6 shows the dependences of dimensional peak pressure $p_{peak}$ on time for several values of the initial pressure amplitude at the center of the drop. The peak pressure at the drop center first increases, reaches a certain maximum, and then decreases. One can see that the increase begins earlier and the maximum peak pressure is higher for higher initial pressure amplitudes. This fact was mentioned above (see Eq. (42)).

## 4. DISCUSSION AND CONCLUSIONS

Our assumption concerning the slowness of harmonic amplitude variation and the corresponding pressure profile is in good agreement with the results of the above analysis. Indeed, for $\Gamma \ll 1$, characteristic amplitude variations occur on scale $\tau_{nl}$. From Eq. (30) it follows that, within time interval $\tau_{nl}$, the number of periods of sound pressure oscillation in the drop is $N = 2\rho_0 c_0^2/(\pi\beta P_0) \gg 1$. For example, for a drop of water at $P_0 = 1$ MPa, the latter estimate gives $N \approx 400$; i.e., the characteristic amplitude variation occurs

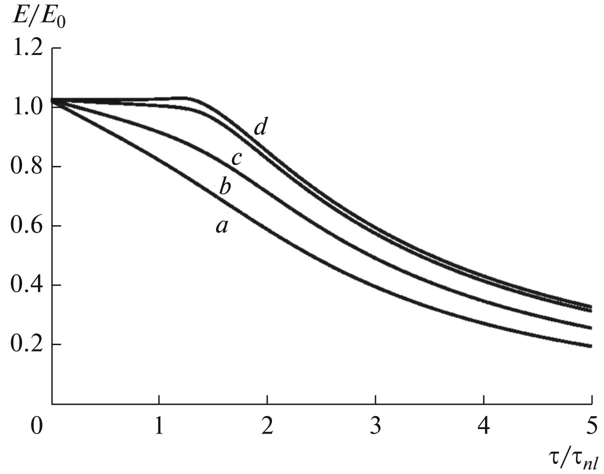


**Fig. 5.** Acoustic energy in the drop, $E$ normalized by its initial value $E_0$ as a function of normalized slow time $\tau/\tau_{nl}$ for different values of parameter $\Gamma$: (*a*) 0.1, (*b*) 0.05, (*c*) 0.01, and (*d*) 0.001.

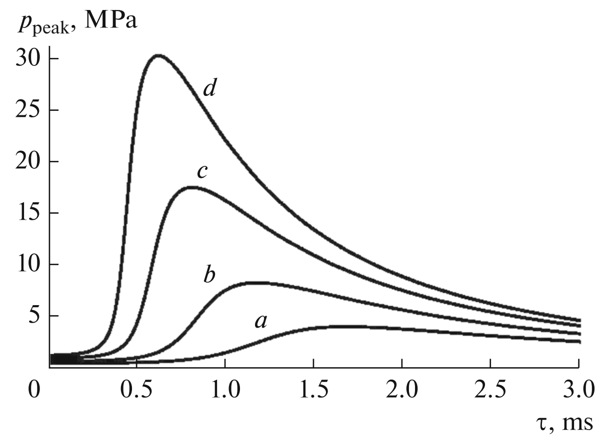


**Fig. 6.** Dependence of peak pressure $p_{peak}$ on slow time $\tau$ in a drop of water with a diameter of 1.5 mm for different values of initial pressure amplitude $P_0$ (MPa): (*a*) 0.5, (*b*) 0.7, (*c*) 1, and (*d*) 1.3.

within several hundreds of oscillation periods, which is definitely very slow.

From the analysis performed above, one can see that evolution of a nonlinear standing wave in a spherical fluid resonator with a perfectly soft boundary has a number of pronounced specific features. Unlike the cases of nonlinear acoustic resonators with rigid walls considered earlier, nonlinear distortion does not lead to formation of shock (step) regions (Fig. 4). The physical reason underlying this feature in the behavior of waves is the inversion of waves at reflection from the resonator surface and at converging-to-diverging wave transformation at the resonator center. Still, the aforementioned inversion cannot suppress nonlinear distortions: in the resonator under study, efficient harmonic generation takes place (Fig. 1). In the course of nonlinear evolution, a standing wave takes the form of an alternately converging–diverging pulse with a sharp peak and with the peak pressure at the resonator center far exceeding the initial wave amplitude (Fig. 4). The maximum amplification of peak pressure occurs at the center of the drop where the time dependence of sound pressure has the form of a periodic sequence of short bipolar pulses. According to Eq. (41), this waveform represents a close approximation of the Hilbert transform of the derivative of a saw-tooth wave described by Burgers equation.

In practice, nonlinear amplification may lead to considerable growth of peak pressure according to estimate (42). For example, in the case of oscillations of a drop in an acoustic fountain considered above, for $P_0 = 1.3$ MPa, within time interval $\tau \approx 0.6$ ms, the peak pressure reaches $p_{max} \approx 30$ MPa (Fig. 6). That high level of negative sound pressure exceeds the ultimate strength of water [23, 24]. This suggests that nonlinear phenomena considered above may be a factor of the instability of drops in an acoustic fountain [18].

In closing, we note an important specific feature of nonlinear evolution of acoustic field in the resonator under study: in a spherical resonator, the energy redistribution within the spectrum (between harmonics), which is typical of nondispersive waves, leads to energy redistribution in space as well. At the same time, despite the total energy decrease, an extremely high energy concentration may occur near the center of the resonator within a certain time interval.

## ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project nos. 17-02-00261-a and 17-54-33034-ONCO-a.

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*Translated by E. Golyamina*