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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

An Exact Solution to the Helmholtz Equation for a Quasi-Gaussian Beam in the Form of a Superposition of Two Sources and Sinks with Complex Coordinates

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Abstract—An exact solution to the Helmholtz equation is proposed. The solution describes a quasi-Gaussian beam with an arbitrary width and has the form of a superposition of sources and sinks with complex coordinates. It is shown that such a beam always lacks a component that propagates against the principal propagation direction. In addition, when the diameter of the beam exceeds the wavelength, the beam becomes directional in the broad sense: the radiation condition is satisfied with respect to the beam waist plane. For the beam under study, expressions for the angular spectrum and the spherical harmonic expansion coefficients are derived.

Keywords: quasi-Gaussian beams, exact solutions to the Helmholtz equation.

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INTRODUCTION

Wave beams with a Gaussian transverse intensity structure (Gaussian beams) are important objects in wave physics. They are of particular use in optics, where the corresponding solution represents the zero-order transverse mode of a laser resonator [1]. Owing to the simple properties of Gaussian beams, they are often used in the theoretical analysis of waves of other origins, e.g., acoustic waves [2]. Gaussian beams arise in the diffraction theory as the solution to the parabolic equation, which describes the true field in the paraxial approximation only. In practical applications concerned with the use of strongly focused fields, such an approach proves to be inexact and it becomes necessary to use simple solutions in the form of beams, which, instead of the parabolic equation, satisfy the Helmholtz equation

$$\Delta p + k^2 p = 0, \qquad (1)$$

Here, p is the complex sound pressure amplitude, $k = \omega/c$ is the wave number, and c is the velocity of sound. In [3, 4], Eq. (1) was shown to have an exact solution representing an almost-Gaussian beam in the vicinity of the axis. The solution had the form of the field of a point source whose axial coordinate was purely imaginary [5]. In the general case, the field of a point source has the form

$$p = \frac{Ae^{ikR}}{R}. \qquad (2)$$

Here, the time dependence is assumed to be $p \sim e^{-i\omega t}$, A is an arbitrary constant, $R = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$ is the distance to the source, and (x_0, y_0, z_0) are the source coordinates. Let us choose $x_0 = y_0 = 0$ and $z_0 = iz_d$, where z_d is a real constant length. Then, in the region where $R \neq 0$, formula (2) represents an exact solution to the Helmholtz equation. Denoting $r_\perp = \sqrt{x^2 + y^2}$, we reduce solution (2) to the form

$$p = A\frac{e^{ik\sqrt{r_\perp^2 + (z - iz_d)^2}}}{\sqrt{r_\perp^2 + (z - iz_d)^2}}. \qquad (3)$$

This solution represents a Gaussian beam when $r_\perp \ll \sqrt{z^2 + z_d^2}$. Indeed, in this case,

$$R = \sqrt{r_\perp^2 + (z - iz_d)^2} = \pm(z - iz_d)\sqrt{1 + \frac{r_\perp^2}{(z - iz_d)^2}} \approx \pm\left[(z - iz_d) + \frac{r_\perp^2}{2(z - iz_d)}\right]. \qquad (4)$$

Note that, formally, both signs before the square root are permissible, yielding exact solutions to Eq. (1). However, to provide unidirectional wave propagation, it is necessary to choose the proper sign. Assuming that the wave propagates from left to right, we choose the upper sign and represent solution (3) in the following approximate form:

$$p \approx \frac{A}{z - iz_d}\exp\left[ik(z - iz_d) + \frac{ikr_\perp^2}{2(z - iz_d)}\right] = \frac{p_0}{1 + i\,z/z_d}e^{ikz}\exp\left(-\frac{r_\perp^2/a^2}{1 + i\,z/z_d}\right), \quad (5)$$

where $p_0 = iAe^{kz_d}/z_d$, and $a = \sqrt{2z_d/k}$. Solution (5) is nothing but a Gaussian beam with the initial wave amplitude p_0 at the axis and with the transverse radius a. The length z_d introduced above is expressed through a as

$$z_d = ka^2/2 \quad (6)$$

and has the meaning of the diffraction convergence length of the beam.

It may seem that solution (3) generalizes the paraxial Gaussian beam (5) to the general case and can therefore be used for describing beams with arbitrary divergence. However, this is not the case. Solution (3) possesses a singularity, which makes it inapplicable to the rigorous description of freely propagating beams. Primarily, it should be noted that, at the point $(r, z_\perp) = (0, z_d)$, the singularity $p \to \infty$ occurs. At the same time, in Eq. (2), $R \to 0$; therefore the corresponding equation is not the homogeneous Helmholtz equation (1), but an equation with a source on the right-hand side. A closer analysis shows that solution (3), which has the form of a directional Gaussian beam in the near-axis region, requires the introduction of a cut along the line $(z = 0,\ \ r_\perp > z_d)$, to guarantee the choice of the necessary branch of the function $\sqrt{r_\perp^2 + (z - iz_d)^2}$ [6, 7]. The wave is actually transmitted through a circular hole with the radius z_d in a screen positioned at $z = 0$. The beam can be considered as a freely propagating one only under the condition that the diameter of the hole is much greater than the beam diameter; i.e., the singularity should be at a large distance from the axis: $z_d \gg a$. In view of Eq. (6), this requirement is equivalent to the condition $ka \gg 2$, such that the beam diameter should greatly exceed the wavelength. Still, under these conditions, the paraxial approximation is valid. Hence, exact solution (3) shows no advantage over approximate solution (5).

To eliminate the aforementioned drawback of solution (3), the introduction of a sink in addition to the point source was proposed in [8], the sink being identical in strength to the source and opposite in sign:

$$p = A \times \frac{\exp\left[ik\sqrt{r_\perp^2 + (z - iz_d)^2}\right] - \exp\left[-ik\sqrt{r_\perp^2 + (z - iz_d)^2}\right]}{\sqrt{r_\perp^2 + (z - iz_d)^2}} \propto \frac{\sin\sqrt{r_\perp^2 + (z - iz_d)^2}}{\sqrt{r_\perp^2 + (z - iz_d)^2}}. \quad (7)$$

This function has no singularities and, hence, is more suitable for describing a freely propagating beam [6, 9]. Its disadvantage in comparison with the field represented by Eq. (3) is the presence of opposite-traveling waves, which only vanish in the wide-aperture approximation: $kz_d \gg 1$.

A QUASI-GAUSSIAN BEAM AS THE SUPERPOSITION OF THE FIELDS OF TWO COMPLEX SOURCES AND SINKS

The purpose of the present study is to describe a more physical exact solution to the Helmholtz equation for a quasi-Gaussian beam. As such a solution, the following expression can be used:

$$p = p_0\frac{z_d}{2\sinh^2(kz_d)}\left[e^{kz_d}\frac{\sin\left(k\sqrt{r_\perp^2 + (z - iz_d)^2}\right)}{\sqrt{r_\perp^2 + (z - iz_d)^2}} - e^{-kz_d}\frac{\sin\left(k\sqrt{r_\perp^2 + (z + iz_d)^2}\right)}{\sqrt{r_\perp^2 + (z + iz_d)^2}}\right]. \quad (8)$$

One can see that Eq. (8) represents a superposition of four solutions of type (3), namely, two source–sink pairs. The factor appearing before the square brackets is adjusted so as to obtain $p = p_0$ at $(z, r_\perp) = (0, 0)$. It is important to note that, in Eq. (8), the functions $\sin R_\pm/R_\pm$, where $R_\pm = \sqrt{r_\perp^2 + (z \mp iz_d)^2}$, have no singularities and no branch points; hence, solution (8) is free of the drawbacks inherent in solution (3). Unlike the previously proposed solution (7), Eq. (8) contains the second source–sink pair, which enables the absence of the wave propagating in the opposite $(-z)$ direction (see below). Note that, when $kz_d = (ka)^2/2 \gg 1$ all three representations given by Eqs. (3), (7), and (8) take the form of solution (5) for a Gaussian beam. The distinctions only manifest themselves for $ka < {\sim}1$. However, precisely this range of values is of interest in describing strongly focused (or strongly divergent) beams. From this point of view, solution (8) seems to be the most attractive one.

Figure 1 shows the distribution of the total sound pressure $\tilde{p}(z, r_\perp, t) = \left(pe^{-i\omega t} + p^* e^{i\omega t}\right)/2$ at the instant $t = 0$ (i.e., Re p) for different values of the parameter ka. In this representation, one can observe the geometry of the wave fronts and the relative characteristic scales of the beam (the wavelength and the waist radius). One can see that, at $ka = 1$, the wave structure barely resembles a directional beam. However, as soon as at $ka = 2$, the directionality is clearly pronounced, although the waist diameter is smaller than the wavelength. As ka increases, the divergence of the beam decreases.

Solution (8) still has a certain disadvantage: strictly speaking, it does not satisfy the condition that the

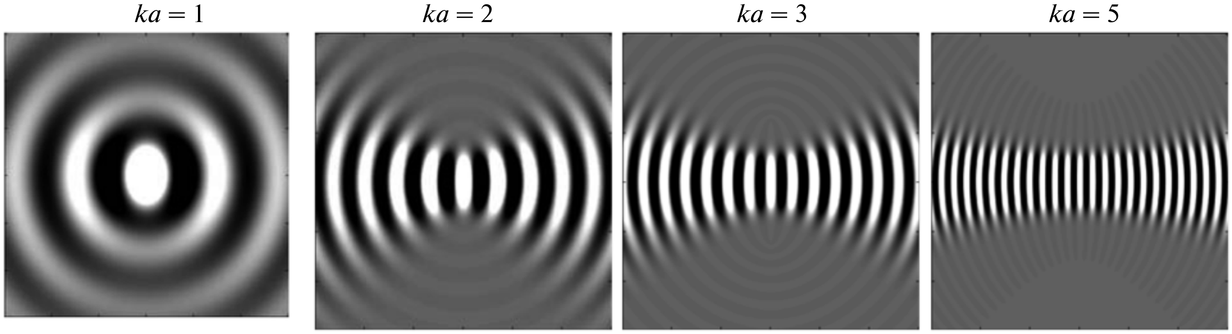

Fig. 1. Instantaneous sound pressure distribution in a quasi-Gaussian beam for a fixed waist radius a and different values of the parameter ka (indicated above the plots). The complex wave amplitude is described by Eq. (8). The z axis is directed from left to right. The shades of grey vary linearly within ±0.2 of the pressure amplitude at the center of the beam waist.

entire wave must propagate from left to right, which is the radiation condition (a similar feature is intrinsic to solution (7)). Indeed, let us consider Eq. (8) at a large distance. Changing to the spherical coordinate system $r_\perp = r\sin\theta$, $z = r\cos\theta$, we obtain $R_\pm|_{r\to\infty} \to r \mp iz_d\cos\theta$, which yields

$$p|_{r\to\infty} = p_0\frac{z_d}{2i\sinh^2(kz_d)} \times\left[\frac{e^{ikr}}{r}\sinh[kz_d(1+\cos\theta)] - \frac{e^{-ikr}}{r}\sinh[kz_d(1-\cos\theta)]\right]. \quad (9)$$

Normalizing the amplitudes of the convergent and divergent waves to the amplitude of the divergent wave at $\theta = 0$, we obtain the following directional patterns for the waves $p_\infty^{(\pm)}(\theta) \sim e^{\pm ikr}/r$:

$$D_\pm(\theta) = \frac{p_\infty^{(\pm)}(\theta)}{p_\infty^{(\pm)}(0)} = \frac{\sinh[kz_d(1\pm\cos\theta)]}{\sinh(2kz_d)}. \quad (10)$$

These dependences are shown in Fig. 2 for different values of ka. For $z > 0$ ($0 \le \theta < 90°$) $D_+(\theta)$ exceeds $D_-(\theta)$; for $z < 0$ ($90° < \theta \le 180°$), the opposite situation takes place. This ensures the directionality of the beam. Along the beam axis, we have $D_+(\pi) = D_-(0) = 0$, so that, the opposite wave is absent. Thus, in terms of solution (8), the directionality of the beam near the beam axis is guaranteed for any ka.

Strictly speaking, the wave field can be considered a directional beam exclusively in case where, for $z < 0$, only the wave of the type e^{-ikr}/r, arriving from infinity is present for any direction while, for $z > 0$, only the receding wave of the type e^{ikr}/r is present. As one can see, for $ka < 3$, this condition fails: a parasitic opposite wave appears. The physical reason for its appearance is evident: if we consider the beam as the radiation of a source in the form of a large spherical bowl, a small beam waist a can only be achieved by increasing the opening angle of the source (i.e., the depth of the bowl) up to complete envelopment of the focusing point.

When ka increases, as soon as it reaches the values $ka \ge 3$, directionality sets in: the parasitic opposite wave almost vanishes. In this case, the function $D_\pm(\theta)$ becomes bell-shaped (i.e., almost Gaussian). Note that the value $ka \approx 3$, which marks the beginning of the region where the parasitic opposite wave is negligibly small, corresponds to the beam radius $a \approx \lambda/2$. In this case, the beam waist is close to the diffraction limit. Thus, solution (8) represents a directional beam up to the point where the wave diameter is identical to the wavelength.

The relative contribution of the parasitic wave can be characterized by the ratio of its power to the power of the principal wave. For definiteness, let us consider the region $z > 0$. Then, according to Eq. (9), in the far zone, the intensities of the principal and parasitic

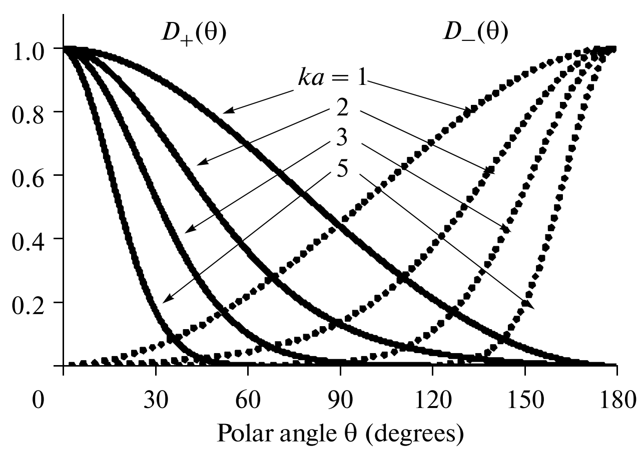

Fig. 2. Directional patterns of the diverging and converging components of solution (8) for different values of the parameter ka.

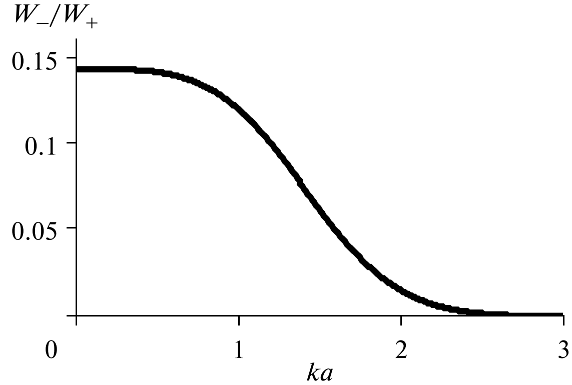

Fig. 3. Power ratio between the opposite and direct components of the beam vs. the parameter ka.

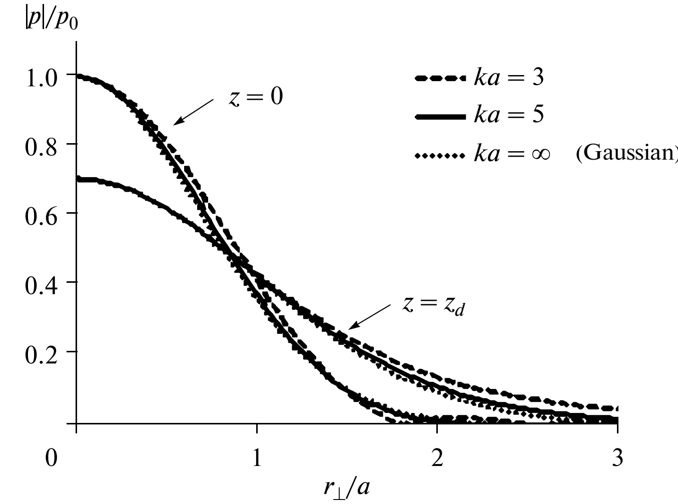

Fig. 4. Transverse distribution of the sound pressure amplitude for different values of ka at the distances $z = 0$ and $z = z_d$.

waves are expressed as $I_\pm|_{r\to\infty} = \frac{1}{2\rho c}\left(\frac{p_0 z_d}{2\sinh^2(kz_d)}\right)^2 \times \frac{\sinh^2[kz_d(1\pm\cos\theta)]}{r^2}$. The full powers of these waves $W_\pm = 2\pi r^2\int_0^{\pi/2} I_\pm \sin\theta d\theta$ are as follows:

$$W_\pm = \pi a^2 \frac{p_0^2}{32\rho c\sinh^4(kz_d)} \times \begin{Bmatrix}\sinh(4kz_d)-\sinh(2kz_d)-2kz_d \\ \sinh(2kz_d)-2kz_d\end{Bmatrix}. \quad (11)$$

Then, the ratio of the powers is

$$\frac{W_-}{W_+} = \frac{\sinh(2kz_d)-2kz_d}{\sinh(4kz_d)-\sinh(2kz_d)-2kz_d}. \quad (12)$$

Figure 3 shows the dependence of W_-/W_+ on the wave size of the waist $ka = \sqrt{2kz_d}$. One can see that the fraction of the parasitic power is only noticeable for $ka < 2$. At $ka = 2$, the fraction of the parasitic power is about 1.5%; for $ka \geq 3$, it is less than 0.01%.

The properties of the beam can also be analyzed on the basis of the velocity and pressure distributions at the beam waist, at $z = 0$. The complex amplitude of the axial component of particle velocity is determined from the equation of motion: $v_z = \frac{1}{ik\rho c}\frac{\partial p}{\partial z}$, where ρ is the density of the medium. According to solution (8), we have

$$p|_{z=0} = p_0\frac{kz_d}{\sinh(kz_d)}\frac{\sinh\left(k\sqrt{z_d^2-r_\perp^2}\right)}{k\sqrt{z_d^2-r_\perp^2}} = p_0\frac{kz_d}{\sinh(kz_d)}\frac{\sin\left(k\sqrt{r_\perp^2-z_d^2}\right)}{k\sqrt{r_\perp^2-z_d^2}}, \quad (13)$$

$$v_z|_{z=0} = -\frac{p_0}{\rho c}\frac{\cosh(kz_d)}{\sinh^2(kz_d)}\frac{(kz_d)^2}{\left(k\sqrt{z_d^2-r_\perp^2}\right)^2} \times\left[\frac{\sinh\left(k\sqrt{z_d^2-r_\perp^2}\right)}{k\sqrt{z_d^2-r_\perp^2}} - \cosh\left(k\sqrt{z_d^2-r_\perp^2}\right)\right] = \frac{p_0}{\rho c}\frac{\cosh(kz_d)}{\sinh^2(kz_d)} \times\frac{(kz_d)^2}{\left(k\sqrt{r_\perp^2-z_d^2}\right)^2}\left[\frac{\sin\left(k\sqrt{r_\perp^2-z_d^2}\right)}{k\sqrt{r_\perp^2-z_d^2}} - \cos\left(k\sqrt{r_\perp^2-z_d^2}\right)\right]. \quad (14)$$

Here, two possible forms of expressions are presented: the first is convenient for $r_\perp \leq z_d$, and the second for $r_\perp > z_d$. One can see that the sound pressure and the axial velocity component are described by real functions so that the plane $z = 0$ coincides with the plane of the phase front. The acoustic power W transmitted through the cross section $z = 0$ is identical to the integral of the intensity $I = \mathrm{Re}\left(pv_z^*\right)/2$ over the cross section area. Using Eqs. (13) and (14), we obtain

$$W = \frac{\pi a^2 p_0^2}{4\rho c}\frac{\cosh(kz_d)}{\sinh(kz_d)}. \quad (15)$$

From Eqs. (11) and (15), we derive

$$W = W_+ - W_-, \quad (16)$$

which should be expected according to the law of conservation of energy.

How much is the term "quasi-Gaussian beam" justified in application to the beam described by Eq. (8)? The answer to this question is given by Fig. 4, where the transverse distribution of the wave amplitude is shown at the beam waist (at $z = 0$) and at the distance identical to the diffraction length ($z = z_d$). One can see that, for $ka = 3$, a small deviation from the exact Gaussian distribution is noticeable, whereas, for large values of ka, the beam profile is barely distinguishable from the Gaussian one.

As it was noticed above, for $ka < 3$, the beam ceases being directional. In the limit $ka \to 0$, Eq. (8) takes the form

$$p\big|_{ka\ll 1} \approx p_0\left\{\frac{\sin kr}{kr} + i\cos\theta\left[\frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}\right]\right\}. \quad (17)$$

From this form, the wave field represents a superposition of monopole and dipole standing waves. The dipole component imparts a certain directionality to the wave but is insufficient to eliminate the standing-wave property. Note that the wave structure described by Eq. (17) does not depend on a. The characteristic size of the waist proves to be on the order of the wavelength and is therefore determined by the diffraction limit.

REPRESENTATION OF THE BEAM AS A SUPERPOSITION OF PLANE WAVES

In calculations, it is sometimes convenient to describe the beam by expanding the solution in plane waves. Any solution to the Helmholtz equation (1) can be represented in the form of a superposition of plane waves with different propagation directions:

$$p(\mathbf{r}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta\, \Pi(\theta,\varphi) e^{i\mathbf{kr}}, \quad (18)$$

where $\Pi(\theta,\varphi)$ characterizes the wave amplitudes and the wave vectors $\mathbf{k}$ are defined by the angles θ and φ of the spherical coordinate system: $\mathbf{k} = k(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Let the radius vector of the observation point be also determined in the spherical coordinate system: $\mathbf{r} = r(\sin\theta_0\cos\varphi_0, \sin\theta_0\sin\varphi_0, \cos\theta_0)$, Then, calculating the integral in the limit $r \to \infty$ by the stationary phase method, we obtain the relation between $\Pi(\theta,\varphi)$ and the behavior of the wave at long distances:

$$p(r,\theta,\varphi)\big|_{r\to\infty} = 2\pi\frac{\Pi(\theta,\varphi)e^{ikr} - \Pi(\pi-\theta,\varphi+\pi)e^{-ikr}}{ikr}. \quad (19)$$

Comparing this equation with asymptotics (9), we obtain

$$\Pi(\theta,\varphi) = p_0\frac{kz_d}{4\pi}\frac{\sinh\left[kz_d(1+\cos\theta)\right]}{\sinh^2(kz_d)} = G(\cos\theta). \quad (20)$$

Owing to the axial symmetry, the function $\Pi(\theta,\varphi) = G(\cos\theta)$ proves to be independent of the polar angle. Therefore, the integral with respect to φ in Eq. (18) yields the Bessel function and the aforementioned expansion takes the form

$$p(r_\perp, z) = 2\pi\int_0^{\pi} d\theta \sin\theta\, G(\cos\theta) J_0(kr_\perp\sin\theta) e^{ikz\cos\theta}. \quad (21)$$

Let us also consider the field representation in terms of the angular spectrum determined in the plane $z = 0$. With respect to this plane, beams with two propagation directions are present: beams propagating to the right and to the left. Hence, the angular spectrum expansion has two components:

$$p(r_\perp, z) = \frac{1}{2\pi}\int_0^{\infty}\left[F_+(k_\perp)e^{iz\sqrt{k^2-k_\perp^2}} + F_-(k_\perp)e^{-iz\sqrt{k^2-k_\perp^2}}\right] \times J_0(k_\perp r_\perp)k_\perp dk_\perp, \quad (22)$$

where $F_+(k_\perp)$ and $F_-(k_\perp)$ are the spectral amplitudes of the direct and opposite beams, respectively. To determine these amplitudes, we rearrange Eq. (21) by separating the integration interval in two parts and introducing the integration variable $k_\perp = k\sin\theta$:

$$p(r_\perp, z) = \frac{2\pi}{k}\int_0^{k}\left[G\left(\frac{\sqrt{k^2-k_\perp^2}}{k}\right)e^{iz\sqrt{k^2-k_\perp^2}} + G\left(-\frac{\sqrt{k^2-k_\perp^2}}{k}\right)e^{-iz\sqrt{k^2-k_\perp^2}}\right]\frac{J_0(k_\perp r_\perp)k_\perp dk_\perp}{\sqrt{k^2-k_\perp^2}}. \quad (23)$$

Comparing Eqs. (22) and (23), we obtain

$$F_\pm(k_\perp) = \begin{cases}\left(\dfrac{2\pi}{k}\right)^2\dfrac{G\left(\pm\sqrt{1-k_\perp^2/k^2}\right)}{\sqrt{1-k_\perp^2/k^2}}, & k_\perp \le k \\ 0, & k_\perp > k\end{cases}. \quad (24)$$

For $k_\perp > k$, both amplitudes $F_+(k_\perp)$ and $F_-(k_\perp)$ are identical to zero. Consequently the inhomogeneous (attenuating) components of the spectrum are absent, as one would expect in view of the absence of sources in the plane $z = 0$. In particular, for the quasi-Gaussian beam under study, under the condition that $0 \le k_\perp \le k$, we obtain

$$F_\pm(k_\perp) = \frac{p_0\pi z_d}{\sinh^2(kz_d)}\frac{\sinh\left[z_d\left(k \pm\sqrt{k^2-k_\perp^2}\right)\right]}{\sqrt{k^2-k_\perp^2}}. \quad (25)$$

Note that, since $v_z = (ik\rho c)^{-1}\partial p/\partial z$, the angular spectra of the axial component of particle velocity are obtained from the pressure spectra $F_\pm(k_\perp)$ by multiplying by $\pm\sqrt{k^2-k_\perp^2}/(k\rho c)$. The dependencies of the spectral velocity amplitudes plotted for the opposite and direct waves $F_\pm(k_\perp)$ within the interval $0 \le k_\perp \le k$ reproduce the plots shown in Fig. 2 for $D_\pm(\theta)$ within the interval $0 \le \theta \le 90°$. Therefore, for $ka \ge 3$, the contribution of the opposite wave to the angular spectrum can be ignored.

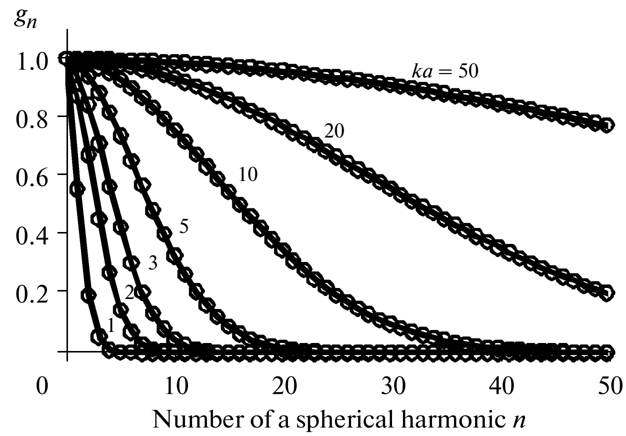

Fig. 5. Normalized spherical harmonic amplitudes g_n (circles) vs. the harmonic number n for different values of ka.

REPRESENTATION OF THE BEAM AS A SUPERPOSITION OF SPHERICAL HARMONICS

In addition to the above expansion of beam (8) in plane waves with different propagation directions, in some cases it may be useful to represent the beam in the form of an expansion in spherical harmonics. To determine the corresponding expansion, let us use the addition theorem for the Bessel functions [10]:

$$\frac{\sin W}{W} = \pi\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)P_n(\cos\theta)\frac{J_{n+1/2}(\xi)}{\sqrt{\xi}}\frac{J_{n+1/2}(\varsigma)}{\sqrt{\varsigma}}, \quad (26)$$

where $W = \sqrt{\xi^2 + \varsigma^2 - 2\xi\varsigma\cos\theta}$ and all the variables are assumed to be complex in the general case; the sign before the root is chosen so that, for $\varsigma \to 0$ the root is positive. In terms of the spherical Bessel functions $j_n(x) = \sqrt{\pi/2x} \times J_{n+1/2}(x)$, Eq. (26) takes the form

$$\frac{\sin W}{W} = \sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\, j_n(\xi) j_n(\varsigma). \quad (27)$$

Let $\xi = kr$, and $\varsigma = ikz_d$. Then, we obtain

$$\frac{\sin\left[k\sqrt{r_\perp^2 + (z \mp iz_d)^2}\right]}{\sqrt{r_\perp^2 + (z \mp iz_d)^2}} = k\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\, j_n(kr) j_n(\pm iz_d). \quad (28)$$

A spherical Bessel function of a purely imaginary argument is expressed through the Infeld function of a real positive argument: $j_n(ix) = i^n\sqrt{\pi/2x}I_{n+1/2}(x)$, $j_n(-ix) = (-1)^{n+1} i^n\sqrt{\pi/2x}I_{n+1/2}(x)$. Taking this into account, we arrive at the desired representation of the quasi-Gaussian beam (8) under study in the form of a spherical harmonic expansion:

$$p = p_0\sum_{n=0}^{\infty} i^n(2n+1)g_n(kz_d)\, j_n(kr)P_n(\cos\theta), \quad (29)$$

where

$$g_n(x) = \frac{1-(-1)^n e^{-2x}}{\left(1-e^{-2x}\right)^2}\varsigma_n(x), \quad (30)$$

$$\varsigma_n(x) = e^{-x}\sqrt{2\pi x}I_{n+1/2}(x). \quad (31)$$

Note that series (29) converges everywhere. The Infeld functions of a half-integer argument are expressed through an exponential function [10]. In particular, for subscripts of the function $\varsigma_n(x)$, we have $\varsigma_0(z) = 1 - e^{-2z}$, $\varsigma_1(z) = 1 - z^{-1} + e^{-2z}\left(1 + z^{-1}\right)$. Other functions can be expressed using the recurrence relation $\varsigma_n(z) = -\varsigma_{n-1}(z)(2n-1)/z + \varsigma_{n-2}(z)$. However, with an increase in n, this algorithm becomes unstable; therefore, in numerical calculations, it is more convenient to use the inverse recurrence algorithm: $\varsigma_{n-1}(z) = \varsigma_n(z)(2n+1)/z + \varsigma_{n+1}(z)$.

The dependence of the relative harmonic amplitudes g_n on their number is shown in Fig. 5 for different values of ka. The coefficients g_n decrease with increasing n by varying from $g_0 = 1$ to approximately zero for $n > 5ka$. This means that, at $ka \sim 1$, in expansion (29) it is sufficient to take into account a relatively small number of series terms, which is convenient for calculations. For wide beams ($ka \gg 1$), with allowance for the asymptotics $I_{n+1/2}(x \gg n) \approx e^x/\sqrt{2\pi x}$ at $x = kz_d \to \infty$, we obtain $g_n(kz_d) \to 1$, which should be expected in the case of a plane wave [10].

Let us consider the behavior of the solution in the form of series (29) in the limit $ka \to 0$. For this purpose, we use the fact that, for small argument values, the Infeld function is expressed as $I_\nu(x)|_{z\to 0} \to (x/2)^\nu/\Gamma(\nu+1)$, where $\Gamma(...)$ is the gamma function [10]. This suggests that $\varsigma_n(x)|_{x\to 0} \to 2x^{n+1}/[1\times 3\times 5\times \ldots\times (2n+1)]$. Therefore, according to Eq. (30), $g_0(0) = 1$, $g_1(0) = 1/6$, $g_{n\geq 2}(0) = 0$, and in the limit $ka \to 0$, series (29) contains only two terms and, as one would expect, takes the form of Eq. (17).

It is also of interest to study the behavior of series (29) in the far field. Note that, since the series coefficients $(2n+1)g_n(kz_d)$ tend to zero when $n \to \infty$, the series actually contains a finite number of terms (see Fig. 5). This means that, when $r \to \infty$, for all the Bessel functions $j_n(kr)$ involved in the series, one can assume that $kr \gg n$, which allows the use of the asymptotics $j_n(x) \approx \mathrm{Re}\left[(-i)^{n+1} e^{ix}\right]/x$. Hence,

$$p|_{r\to\infty} = \\ = \frac{p_0}{2ikr}\sum_{n=0}^{\infty}(2n+1)\,g_n\left[e^{ikr} - (-1)^n e^{-ikr}\right]P_n(\cos\theta). \quad (32)$$

Comparing the resulting formula with Eq. (19), we obtain the relation of the plane wave amplitudes $\Pi(\theta,\varphi) = G(\cos\theta)$ appearing in expansion (18) to the coefficients g_n involved in spherical harmonic expansion (29):

$$G(\cos\theta) = \frac{p_0}{4\pi}\sum_{n=0}^{\infty}(2n+1)\,g_n P_n(\cos\theta). \quad (33)$$

Since this relation is derived without using the explicit expression for g_n, it is valid for an axisymmetric beam of an arbitrary form. Using the orthogonality of the Legendre polynomials $\int_{-1}^{1} P_n(x)\,P_l(x)\,dx = 2\delta_{nl}/(2n+1)$, it is possible to invert Eq. (33), to express the coefficients g_n through the directionality function $G(\cos\theta)$:

$$g_n = \frac{2\pi}{p_0}\int_{-1}^{1} G(x)\,P_n(x)\,dx. \quad (34)$$

For example, if $G(\cos\theta)$ is preset in the form of Eq. (20), the coefficients g_n will be determined by Eqs. (30) and (31). This can be easily verified by using the tabulated integral $\int_{-1}^{1} e^{-bx} P_n(x)\,dx = (-1)^n\sqrt{2\pi/b}\, \times I_{n+1/2}(b)$ [11].

In a similar way, it is possible to construct other beams. In particular, for the class of directional beams, one obtains $G(x) \equiv 0$ for $-1 \le x < 0$. In this case, $F_-(k_\perp) = 0$; all the possible plane waves propagate to the right and are described by the angular wave spectrum $F_+(k_\perp)$. Using Eq. (24), we obtain the expression that relates the spherical harmonic expansion coefficients g_n appearing in Eq. (29) to the angular spectrum $F_+(k_\perp)$:

$$g_n = \frac{k^2}{2\pi p_0}\int_0^1 F_+\left(k\sqrt{1-x^2}\right)P_n(x)\,x\,dx. \quad (35)$$

Equation (29) with coefficients (35) may be convenient in solving a number of problems, for example, in analyzing the scattering from spherically symmetric objects [12].

CONCLUSIONS

The proposed exact solution to the Helmholtz equation (8) describes quasi-Gaussian beams for which the waist diameter is comparable to the wavelength. The solution represents the superposition of two point sources and two point sinks with complex coordinates. It is shown that, in such a beam, the component propagating against the principal direction is always absent; in addition, when the diameter of the beam waist exceeds the wavelength, the beam becomes directional in the broad sense: the power fraction of the parasitic opposite wave becomes negligibly small. Expressions for the angular spectrum and the spherical harmonic expansion coefficients are derived for the beam under study.

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