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NONLINEAR ACOUSTICS

Determination of the Exact Solutions to the Inhomogeneous Burgers Equation with the Use of the Darboux Transformation

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Abstract—A method of determining the exact solutions to the Burgers equation on the basis of the Darboux transformation is described. It is shown that a single application of the Darboux transformation to the homogeneous Burgers equation transforms the latter into the inhomogeneous equation describing acoustic wave propagation against transonic flow in the de Laval nozzle. In this case, the contraction ratio of the nozzle is fixed and determined by the viscosity coefficient of the medium. Based on the exact solution of the homogeneous Burgers equation, for the aforementioned problem of the flow in the nozzle, all the possible regular steady-state solutions are presented and the evolution of nonstationary solutions is investigated. The algorithm of a multiple Darboux transformation, which allows an increase in the strength of inhomogeneity, i.e., in the contraction ratio of the nozzle, is determined. This approach leads to a discrete set of possible contraction ratios at which exact solutions can be obtained. The Crum's theorem is used to derive a formula that allows determination of the exact solutions to the inhomogeneous Burgers equation from the solutions to the homogeneous heat transfer equation. It is noted that, in fact, the proposed algorithm of the multiple Darboux transformation makes it possible to decrease the viscosity coefficient of the medium in a discrete way.

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INTRODUCTION

The Burgers equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \delta\frac{\partial^2 u}{\partial x^2} = 0, \qquad (1)$$

is one of the reference nonlinear equations of mathematical physics. Initially, it was formulated as a model equation for describing one-dimensional turbulence [1]; later, it was also found to describe a number of physical phenomena of different natures. In particular, in nonlinear acoustics, the Burgers equation was used to describe the propagation of one-dimensional finite-amplitude acoustic waves in the presence of dissipation. In this case, $u(x,t)$ determines the hydrodynamic particle velocity as a function of the x coordinate and time t, while the constant δ characterizes the viscosity and thermal conductivity of the medium [2, 3].

In the presence of sources in the medium, the Burgers equation is modified and becomes an inhomogeneous equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \delta\frac{\partial^2 u}{\partial x^2} = F, \qquad (2)$$

where the function $F(x,t)$ appearing on the right-hand side describes the sources. This equation was first derived from the hydrodynamics equations and used in studying the laser generation of sound [4, 5]. Later on, it was applied to other physical situations [6]. Approximately at the same time, the inhomogeneous Burgers equation was analyzed from the mathematical point of view [7–10].

Equation (1) is unique in that, through the substitution

$$u = -\frac{2\delta}{w}\frac{\partial w}{\partial x} \qquad (3)$$

it can be reduced to the linear heat transfer equation for the auxiliary function $w(x,t)$. This linearization makes it possible to obtain the general solution to the Burgers equation. Transformation (3) for the homogeneous equation was determined in [11]; however, it has become more popular after the appearance of publications [12, 13] and, therefore, has been called the Hopf–Cole substitution. Linearization makes it possible to represent the solution to the initial problem in integral form. For an arbitrary initial profile $u(x,0)$, the corresponding integral has not been analytically calculated. Nevertheless, for a number of specific cases, such a calculation is possible. This approach allows regular determination of the exact solution to the Burgers equation for a number of physically interesting situations [14].

It is remarkable that the Hopf–Cole substitution (3) also linearizes inhomogeneous equation (2). This fact was first noticed in [5]. The corresponding linear equation has the form

$$\frac{\partial w}{\partial t} - \delta \frac{\partial^2 w}{\partial x^2} + Vw = 0, \qquad (4)$$

where the "potential" $V(x,t)$ is determined by the sources

$$F = 2\delta \frac{\partial V}{\partial x}. \qquad (5)$$

For brevity, we call Eq. (4) the heat transfer equation by analogy with the case of $V = 0$. As applied to the problem of heat transfer, the additional term Vw describes the internal heat removal (or heat supply) proportional to the local temperature. A similar situation is typical of, e.g., the heat transfer process in biological tissues containing numerous small blood vessels [15]. We note that Eq. (4) belongs to the class of equations of the type of

$$\alpha \frac{\partial \psi}{\partial t} - \delta \frac{\partial^2 \psi}{\partial x^2} + V\psi = 0, \qquad (6)$$

which are solved by the Darboux transformation method [16]. In particular, at $\alpha = i$, Eq. (6) represents a nonstationary Schrödinger equation for the wave function ψ of a particle performing a one-dimensional motion in the potential field V. It is precisely in application to the Schrödinger equation that the Darboux transformation has attracted the attention of researchers in the last few years. This approach provides a possibility of obtaining new solvable models of quantum mechanics; it is also of interest from the point of view of studying the inverse scattering problem and the theory of solitons (see, e.g., [17]).

THE DARBOUX TRANSFORMATION

For an equation of the type of Eq. (6), the Darboux transformation represents the following linear transformation of the wave function [16]:

$$\tilde{\psi}(x,t) = [R(x,t) + \partial/\partial x]\psi(x,t). \qquad (7)$$

The characteristic feature of this transformation consists in that, if the function ψ satisfies Eq. (6) with the potential V, the function $\tilde{\psi}$ also satisfies Eq. (6) with another potential $\tilde{V}$ in the case of an adequately chosen dependence $R(x,t)$. This property is the basis of the Darboux transformation method. The idea is as follows: for a known solution of the equation with a "simple" potential (e.g., $V = 0$), we choose the function R so as to obtain the solution to the equation with a more "complicated" potential.

Let us consider the aforementioned method in application to Eq. (4). We set $s = 2\delta R$ in Eq. (7) and consider the Darboux transformation for solutions (3):

$$\tilde{w}(x,t) = \left[s(x,t)/(2\delta) + \frac{\partial}{\partial x} \right] w(x,t). \qquad (8)$$

Calculating the derivatives $\partial \tilde{w}/\partial t$ and $\partial^2 \tilde{w}/\partial x^2$ from Eq. (8) and taking into account Eq. (4), we obtain that, if the auxiliary function $s(x,t)$ satisfies the equation

$$\frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} - \delta \frac{\partial^2 s}{\partial x^2} = F \qquad (9)$$

with right-hand side (5), the new function $\tilde{w}(x,t)$ represents the solution to an equation of the type of Eq. (4) with the new potential $\tilde{V} = V + \partial s/\partial x$:

$$\frac{\partial \tilde{w}}{\partial t} - \delta \frac{\partial^2 \tilde{w}}{\partial x^2} + \tilde{V}\tilde{w} = 0. \qquad (10)$$

Condition (9) imposed on the function $s(x,t)$ is nothing but inhomogeneous Burgers equation (2), the solutions to which are related through Hopf–Cole transformation (3) to the solutions of the corresponding heat transfer equation (4). This fact is important; it suggests that the relation between the Burgers equation and the heat transfer equation is more profound than the simple change from one equation to another through the Hopf–Cole transformation (although the existence of transformation (3) reducing the nonlinear equation to the linear one is unique by itself).

From the equations given above, one can see that, for both the inhomogeneous Burgers equation and the related heat transfer equation, the Darboux transformation can be represented in a form corresponding to only one of these two equations.

For example, let us consider heat transfer equation (4) for the function $w(x,t)$. Let $v(x,t)$ be one of its particular solutions. Then, for the function

$$\tilde{w} = w2\delta \frac{\partial}{\partial x}(\ln v - \ln w)$$

we obtain an equation similar to Eq. (4) but with the new potential

$$\tilde{V} = V - 2\delta \partial^2 (\ln v)/\partial x^2.$$

In the same way, we can deal with Eq. (2). If we take one of the solutions $s(x,t)$ to Burgers equation (2) with right-hand side (5), the substitution

$$\tilde{u} = u - 2\delta \frac{\partial}{\partial x} \ln(u - s) \qquad (11)$$

will again lead us to the inhomogeneous Burgers equation, but with a new right-hand side:

$$\tilde{F} = F + 2\delta \partial^2 s/\partial x^2. \qquad (12)$$

In the particular case of $s = \text{const}$, the Darboux transformation does not affect the right-hand side, i.e., $\tilde{F} = F$, and, for $F = 0$, Eq. (11) coincides with the formula of the Backlund autotransformation of the homogeneous Burgers equation [18]. Presumably, formulas (11) and (12) first appeared in [19], where they were used to analyze the solutions to inhomogeneous Burgers equations with right-hand sides obtained on the basis of the class of exact solutions to the inhomogeneous Burgers equation with the right-hand side linear in x.

The aforementioned procedure of changing from the Burgers equation with the source F to the equation with another source $\tilde{F}$ (or, which is equivalent, the passage from the problem of heat transfer with one potential V to the problem with the other potential $\tilde{V}$) can be repeated. Multiple application of the Darboux transformation can, in principle, allow one to find the solutions to the inhomogeneous Burgers equation for a set of nontrivial sources. In the present paper, we consider the chain of solutions to Eq. (4) with different potentials obtained by multiple application of the Darboux transformation under the assumption that the initial potential is $V = 0$. By applying the Hopf–Cole transformation (3), from the aforementioned solutions of Eq. (4), we determine a set of exactly solvable inhomogeneous Burgers equations (2). Below, we show that the solutions obtained in this way describe the nonlinear wave propagation against the transonic viscous liquid flow in a de Laval nozzle for nozzles with different contraction ratios.

SOLUTIONS TO THE INHOMOGENEOUS BURGERS EQUATION OBTAINED BY SINGLE APPLICATION OF THE DARBOUX TRANSFORMATION TO THE HOMOGENEOUS EQUATION

We limit our consideration to the class of steady-state inhomogeneities; i.e., we assume that the right-hand side of the inhomogeneous Burgers equation does not depend on time t.

We proceed from the homogeneous Burgers equation (1). Its solutions are well known and expressed through the solutions to the classical heat transfer equation with the help of Hopf–Cole substitution (3). As we mentioned above, the inhomogeneous Burgers equation "generated" by the Darboux transformation has the form

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial x} - \delta\frac{\partial^2 \tilde{u}}{\partial x^2} = \tilde{F}, \tag{13}$$

where $\tilde{F}$ is given by Eq. (12), with $s(x,t)$ being represented by a solution to the homogeneous Burgers equation (1). If we consider the case of time-independent sources, the quantity s can represent a stationary solution. Since all the physical sources are assumed to be regular in space, such stationary solutions have the form $u = s_0 = -(2\delta/L)\tanh[(x-x_0)/L]$, where the constant L has the meaning of the characteristic width of the transition region in the smoothed step function and x_0 is the coordinate of the center of this region. Without loss of generality, we set $L = 1$ and $x_0 = 0$, because a more general situation is reduced to a simple change of variables. Thus, we set

$$s_0(x) = -2\delta\tanh x. \tag{14}$$

Note that, when solution (14) is used, the inhomogeneous Burgers equation with the right-hand side

$$\tilde{F} = 2\delta\, d^2 s_0/dx^2 = -4\delta^2\frac{d}{dx}\left(\cosh^{-2} x\right) \tag{15}$$

describes the propagation of a nonlinear acoustic wave against the transonic flow in a pipe containing a segment with a decreasing cross section (the de Laval nozzle). Indeed, one-dimensional propagation of waves in a pipe with a smoothly varying cross section $A(x)$ is described by the inhomogeneous equation of the type of Eq. (2) with the right-hand side $F \sim dA/dx$ [20]. Therefore, function (15) yields the dependence $A(x) \sim \mathrm{const} - \cosh^{-2} x$, which describes the smooth contraction–expansion in the vicinity of $x = 0$ within the region characterized by the length $\Delta x \sim 1$. However, we note that the resulting inhomogeneity strength $\tilde{F}$ given by Eq. (15) is not arbitrary: it is proportional to the square of the dissipation coefficient δ and, therefore, describes the nozzle adjusted to the preset viscosity coefficient of the medium. Nevertheless, it is reasonable to assume that the chosen particular case reflects the main characteristic features of the wave evolution in a de Laval nozzle of arbitrary form. This assumption will be justified below by considering a chain of inhomogeneities with increasing amplitude.

We consider the possible form of steady-state solutions to the aforementioned inhomogeneous equation. They can be found using Eq. (11), where $s = s_0$, i.e., is given by Eq. (14), while u should represent other possible steady-state solutions to Burgers equation (1). In the general case, the steady-state solutions to the homogeneous Burgers equation fall into three classes: (i) constant solutions $u = U_0 = \mathrm{const}$; (ii) solutions in the form of a smooth transition from $u(-\infty) = U_\infty$ to $u(+\infty) = -U_\infty$ with the center at $x = x_0$: $u = -U_\infty\tanh[U_\infty(x-x_0)/(2\delta)]$ (solution (14) is precisely of this type); and (iii) singular solutions in the form of a hyperbolic cotangent: $u = -U_\infty/\tanh[U_\infty(x-x_0)/(2\delta)]$. Note that, in analyzing the solutions to Burgers equation (1), solutions of types (i) and (iii) are usually not considered, because they are trivial and nonphysical, respectively. However, for our considerations, precisely these solutions prove to be useful. As for the regular solution of type (ii), it is only at $U_\infty = 2\delta$ and nonzero values of x_0 that this solution does not intersect curve (14); therefore, its substitution in Eq. (11) leads to a singularity and, hence, the corresponding solution is nonphysical.

When $u = U_0$, the singularity in Eq. (11) is eliminated if $|U_0| > 2\delta$. In this case, a family of steady-state

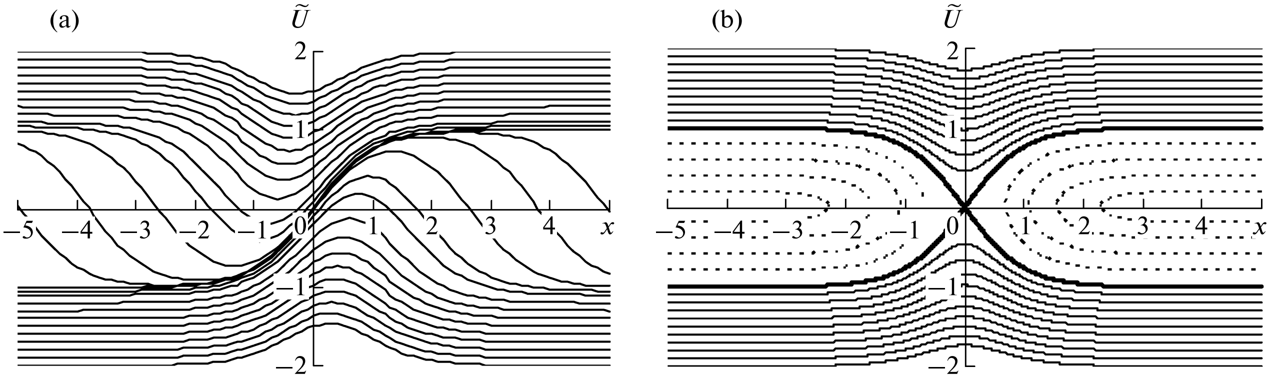

Fig. 1. Steady-state solutions to the inhomogeneous equation with the right-hand side described by Eq. (15). The corresponding solutions describe the velocity of the transonic flow in the de Laval nozzle. The flow is directed from left to right; the quantity $\tilde{U}$ is proportional to the Mach number of the flow; i.e., when $\tilde{U} < 0$, the flow is subsonic, and when $\tilde{U} > 0$, the flow is supersonic. (a) The case of a viscous medium and (b) the case of a nonviscous medium. The dashed lines indicate the two-valued steady-state solutions that are physically nonrealizable. The thick separatrices correspond to the transonic flow regime.

solutions is obtained for the inhomogeneous Burgers equation. In the dimensionless form, the corresponding dependences (11) take the form

$$\tilde{U} = \alpha - \frac{\cosh^{-2} x}{\alpha + \tanh x}, \tag{16}$$

where $\alpha = U_0/(2\delta)$ is the parameter of the family, $|\alpha| \geq 1$, and $\tilde{U} = \tilde{u}/(2\delta)$. Curves (16) are displayed in Fig. 1a. For comparison, in Fig. 1b we present the corresponding dependences for the dimensionless velocity of the stationary flow in the de Laval nozzle in the absence of dissipation of the medium (this case was studied in [20, 21]). Omitting the term $\sim \partial^2\tilde{u}/\partial x^2$ in Eq. (13), for the steady-state solution in a medium without dissipation, we obtain the following expression in the dimensionless coordinates: $\tilde{U} = \pm\sqrt{\beta^2 - \cosh^{-2} x}$, where β is the parameter of the family.

From comparison of Figs. 1a and 1b, we can estimate the effect of the viscosity of the medium on the nature of the steady-state flow.

In the absence of viscosity, three flow regimes are clearly distinguished: subsonic ($\tilde{U} < 0$), supersonic ($\tilde{U} > 0$), and transonic. In the subsonic regime, the flow velocity $\tilde{U}$ increases in the vicinity of the critical (minimal) cross section; in the supersonic regime, the flow velocity decreases. The transonic regime, which is illustrated by thick lines in Fig. 1b, can be of either accelerated (as x increases, the quantity $\tilde{U}$ grows) or decelerated type ($\tilde{U}$ decreases). One can see the clearly defined region bounded by the transonic separatrices, within which no stationary flow lines can be observed.

In the case of nonzero viscosity (Fig. 1a), the steady-state regimes also fall into the three aforementioned types. However, in this case, the transonic regime is represented not by two separatrices, but by the whole family of curves, and the corresponding flows, which are subsonic away from the inhomogeneity region, become supersonic only in a certain region to the right of the critical cross section. In a similar way, the flows that are supersonic in the region $|x| \gg 1$ become locally subsonic immediately before the critical cross section. Thus, viscosity qualitatively changes the nature of the transonic flow.

The aforementioned steady-state solutions to Eq. (13) with right-hand side (15) represent one of the two possible classes of solutions. As was noted above, the second class is obtained from the singular solutions to the homogeneous Burgers equation: $u = -U_\infty/\tanh[U_\infty(x - x_0)/(2\delta)]$. In this case, from Eqs. (11) and (14), for $\tilde{U} = \tilde{u}/(2\delta)$ we obtain a two-parameter family of curves:

$$\tilde{U} = \frac{\gamma \tanh x + (\cosh^{-2} x - \gamma^2)\tanh[\gamma(x - x_0)]}{\gamma - \tanh x \tanh[\gamma(x - x_0)]}, \tag{17}$$

where the constants $\gamma = U_\infty/(2\delta)$ and x_0 are parameters such that $\gamma \geq 1$ and x_0 is an arbitrary number.

Figure 2 shows the characteristic steady-state profiles for different parameters γ and x_0. Note that, for $x \ll x_0$, we have $\tanh[\gamma(x - x_0)] \approx -1$; then, from Eq. (17) we obtain formula (16): $\tilde{U} \approx \alpha - \cosh^{-2} x/(\alpha + \tanh x)$, where $\alpha = \gamma$. In the same way, for $x \gg x_0$, we again obtain formula (16) $\tilde{U} \approx \alpha - \cosh^{-2} x/(\alpha + \tanh x)$, where $\alpha = -\gamma$. Thus, the steady-state solutions of type (17) describe the solutions in the form of the transition between the supersonic ($\alpha = \gamma$) and subsonic ($\alpha = -\gamma$) profiles of family (16). The coordinate of the center of this tran-

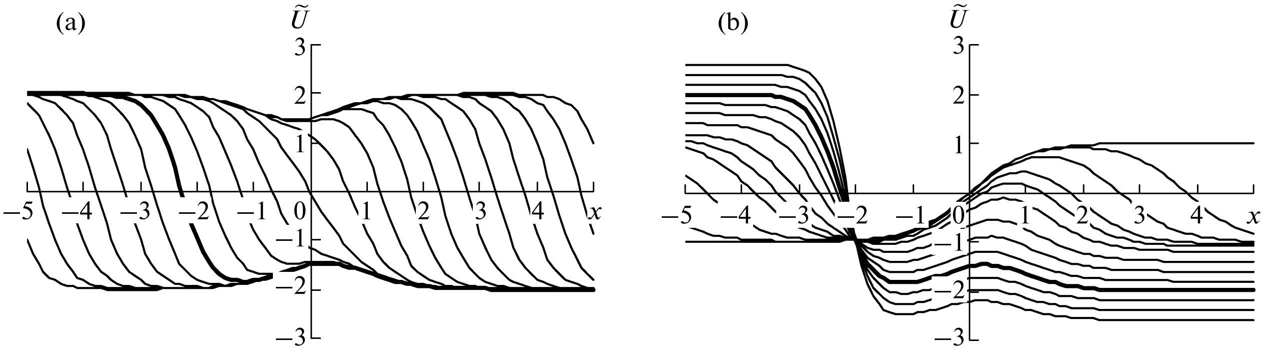

Fig. 2. Two-parameter family of steady-state solutions to the inhomogeneous equation; the family is described by Eq. (17): (a) $\gamma = 2$, x_0 takes on different values, and the thick line corresponds to the case of $x_0 = -2$; (b) $x_0 = -2$, γ takes on different values, and the thick line corresponds to the case of $\gamma = 2$.

sition is given by the parameter x_0. In the case of a nonviscous medium, the corresponding stationary profiles can be constructed as weak solutions to the corresponding inhomogeneous equation by introducing the jumplike transition at $x = x_0$ from the supersonic stationary flow occurring in the region $x < x_0$ to the subsonic stationary flow in the region $x > x_0$.

In addition to the regular steady-state solutions described above, singular steady-state solutions take place by analogy with the case of the homogeneous Burgers equation, which, in addition to the smooth solutions in the form of the hyperbolic tangent, has solutions with a singularity in the form of a hyperbolic contangent. Such singular solutions of the inhomogeneous Burgers equation are nonphysical and, therefore, not analyzed in this paper. However, in the case of multiple application of the Darboux transformation (see below), the singular steady-state solutions are also of interest, because, after their substitution in Eq. (11), they allow us to obtain smooth solutions. The above example of obtaining family (17) illustrates this possibility.

Let us use the Darboux transformation technique to obtain nonstationary solutions of Eq. (13) with right-hand side (15). As an example, we consider a wave that arrives at the region of inhomogeneity and has the form of a steplike transition between the steady-state solutions shown in Fig. 1a. Remember that these steady-state solutions were obtained by substituting the steady-state solution of the homogeneous Burgers equation $u = U_0$ in the Darboux transformation (11). Now, as the generating solution u, we consider the nonstationary solution to Eq. (1) in the form of the transition that is smoothed by viscosity and occurs from the stationary flow with $u(\infty) = U_0 - A/2$ to the other stationary flow with $u(\infty) = U_0 + A/2$:

$$u = U_0 - \frac{A}{2}\tanh\left[\frac{A}{4\delta}(x - x_0 - U_0 t)\right]. \qquad (18)$$

Applying the Darboux transformation (11), we derive the desired expression for $\tilde{U} = \tilde{u}/(2\delta)$:

$$\tilde{U} = \alpha - \Delta\alpha\tanh[\Delta\alpha(x - x_0 - 2\alpha\delta t)] + \frac{\Delta\alpha^2\cosh^{-2}[\Delta\alpha(x - x_0 - 2\alpha\delta t)] - \cosh^{-2}x}{\alpha - \Delta\alpha\tanh[\Delta\alpha(x - x_0 - 2\alpha\delta t)] + \tanh x}, \qquad (19)$$

where $\Delta\alpha = A/(4\delta)$. According to Eqs. (3) and (18), for solution (19) the generating function is $w = \exp\left[-\alpha x + \delta\left(\alpha^2 + \Delta\alpha^2\right)t\right]\cosh[\Delta\alpha(x - x_0 - 2\alpha\delta t)]$. The evolution of the wave described by Eq. (19) is illustrated in Fig. 3. The steady-state solutions corresponding to the two levels of the initial steplike perturbation are indicated by the dashed lines and plotted according to Eq. (16). The initial perturbation is preset to the right of the inhomogeneity region; with the passage of time, it moves leftward and transforms the flow from the initial steady-state regime to the new regime.

The perturbations that have the form of a jumplike transition between the supersonic ($\alpha > 1$) steady-state solutions are analyzed in a similar way. Since Eq. (13) with right-hand side (15) does not change its form under the change of variables $\tilde{u} \to -\tilde{u}$, $x \to -x$, it is possible to apply the results of the above analysis to the case $\alpha < -1$.

MULTIPLE APPLICATION OF THE DARBOUX TRANSFORMATION

We note that, by means of the Darboux transformation, each of steady-state solutions (16) and (17) can be used to obtain inhomogeneous equations with new right-hand sides and the solutions of these new equations will be expressed in a certain way through the solutions of the initial equation. This procedure can be multiply repeated.

Let us consider a chain of inhomogeneous Burgers equations obtained one from another as a result of repeated application of the Darboux transformation.

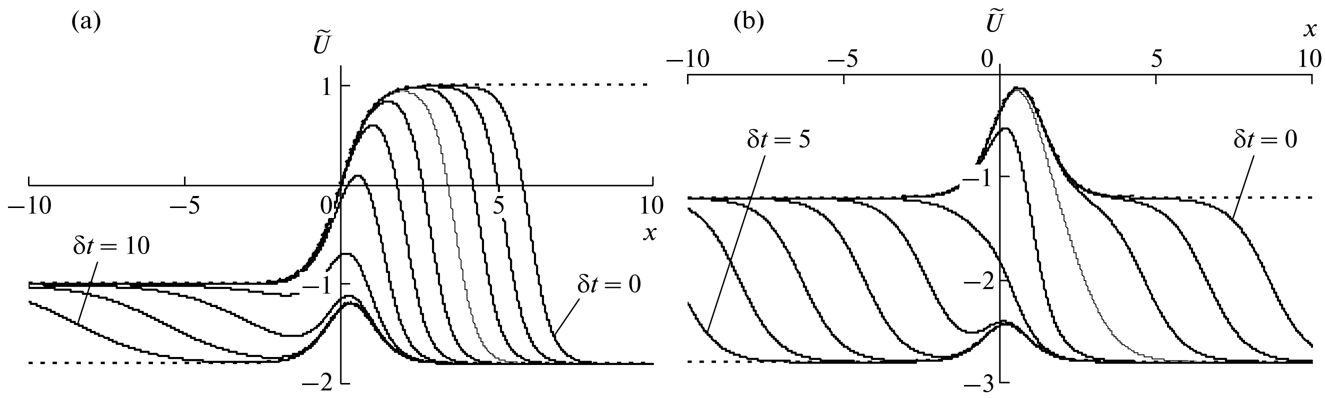

Fig. 3. Passage of the steplike perturbation through the inhomogeneity region (the transformation of the flow in the de Laval nozzle). The wave profiles are shown for successive instants of time. (a) The case of the transonic-to-subsonic flow transformation: $\alpha = -1.4$, $\Delta\alpha = 0.4$, $x_0 = 20$, and $\delta t = 0, 1, \ldots, 10$. (b) The case of the transformation of one subsonic flow regime to another: $\alpha = -2$, $\Delta\alpha = 0.8$, $x_0 = 10$, and $\delta t = 0, 0.5, 1, \ldots, 5$.

We use the subscript $n = 0, 1, 2, \ldots$ to indicate the solution and the right-hand side at the nth step of this procedure:

$$\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} - \delta \frac{\partial^2 u_n}{\partial x^2} = F_n(x). \tag{20}$$

As the initial equation ($n = 0$), we take the homogeneous Burgers equation; i.e., we set $F_0 = 0$. According to transformation (12), after multiple application of the Darboux transformation, the right-hand side takes the form

$$\begin{aligned} F_n &= F_{n-1} + 2\delta \frac{d^2 s_{n-1}}{dx^2} = \ldots \\ &= F_0 + 2\delta \frac{d^2 s_0}{dx^2} + \ldots + 2\delta \frac{d^2 s_{n-1}}{dx^2}, \end{aligned} \tag{21}$$

where $s_n(x)$ is one of the steady-state solutions to Eq. (20). According to Eq. (14), $s_0(x) = -2\delta \tanh x$ and, at the first step, formula (21) is reduced to Eq. (15): $F_1 = 2\delta d^2 s_0/dx^2$. As $s_1(x)$, we can take any of the steady-state solutions of form (16) or (17); i.e., at the very first step, the Darboux transformation generates a multiparameter family of possible sources.

Of the multitude of possible inhomogeneities $F_n(x)$, we consider those describing the given (aforementioned) spatial profile of the de Laval nozzle for every n: $F_n(x) = \Lambda_n d^2 s_0/dx^2$, where $\Lambda_n > 0$ is a certain factor increasing with the number n. This is only possible if, for any n, Eq. (20) has a steady-state solution of the form $s_n = \lambda_n s_0$, where λ_n is the scale factor and $\lambda_0 = 1$. We substitute this form of s_n in Eq. (21) and consider the steady-state solutions to Eq. (20) with the corresponding right-hand sides for $n \geq 1$:

$$\lambda_n^2 s_0 \frac{ds_0}{dx} - \delta\lambda_n \frac{d^2 s_0}{dx^2} = F_n, \tag{22}$$

$$F_n = (1 + \lambda_1 + \ldots + \lambda_{n-1}) 2\delta \frac{d^2 s_0}{dx^2}. \tag{23}$$

With allowance for the fact that $s_0\, ds_0/dx = \delta d^2 s_0/dx^2$, from Eq. (22) we obtain $\lambda_n^2 - \lambda_n = 2(1 + \lambda_1 + \ldots + \lambda_{n-1})$, which, in view of the requirement that F_n should increase with increasing number, leads to the recurrence formula for determining λ_n: $\lambda_n = \left(1 + \sqrt{1 + 8(1 + \lambda_1 + \ldots + \lambda_{n-1})}\right)/2$, $n \geq 1$. This suggests that $\lambda_n = n + 1$. Hence, solutions of the desired form do exist; in addition,

$$s_n = (n+1) s_0 = -2\delta(n+1)\tanh x. \tag{24}$$

For example, $s_1 = 2s_0$ is the solution belonging to family (17) and corresponding to the parameters $\gamma = 2$ and $x_0 = 0$. The sequence of steady-state solutions (24) determines the algorithm of the increase in the “strength” of inhomogeneity with conservation of its spatial profile:

$$F_n(x) = -2\delta^2 n(n+1) \frac{d}{dx}\left(\cosh^{-2} x\right). \tag{25}$$

From Eq. (25) one can see that, by multiply applying the Darboux transformation, it is in principle possible to model the inhomogeneity of arbitrary strength and, thus, make it independent of the viscosity coefficient. A specific feature of this procedure is the discrete character of the possible increase in the source strength: $F_n(x) = F_1(x) n(n+1)/2$. In the actual problem of the de Laval nozzle, the viscous term is usually small; i.e., the jumps $F_{n+1} - F_n$ (the “quanta” of inhomogeneity) are also small.

The increase in the right-hand side (Eq. (25)) at a fixed nozzle profile can also be interpreted as the transition to a less viscous medium. Indeed, if we denote $\hat{\delta} = \delta\sqrt{n(n+1)/2}$, Eq. (20) takes the form

$$\frac{\partial u_n}{\partial t} + u_n \frac{\partial u_n}{\partial x} - \frac{\hat{\delta}}{\sqrt{n(n+1)/2}} \frac{\partial^2 u_n}{\partial x^2} = -4\hat{\delta}^2 \frac{d}{dx}\left(\cosh^{-2} x\right), \quad (26)$$

i.e., the n-fold Darboux transformation leads to the problem of wave propagation in the same nozzle but in a medium the viscosity coefficient of which is smaller by a factor of $\sqrt{n(n+1)/2}$.

DETERMINATION OF EXACT SOLUTIONS TO THE PROBLEM OF FLOW IN THE DE LAVAL NOZZLE FOR DIFFERENT VISCOSITY COEFFICIENTS

Instead of the sequence of Eqs. (20), we consider a chain of equations of the type of Eq. (4), the solutions of which are related through the Hopf–Cole transformation to the solutions of the Burgers equation. Let the n-fold Darboux transformation yield the following equation with the potential V_n:

$$\frac{\partial w_n}{\partial t} - \delta \frac{\partial^2 w_n}{\partial x^2} + V_n w_n = 0. \quad (27)$$

The right-hand side F_n of inhomogeneous Burgers equation (20) is expressed through the potential V_n according to Eq. (5): $F_n = 2\delta \partial V_n/\partial x$. The functions w_n and u_n are related through the Hopf–Cole transformation (3). Remember that, in the case under consideration, we have $V_0 = 0$.

The solutions w_n obtained at the nth step of the multiple Darboux transformation can be expressed in explicit form through the set of solutions of the initial equation alone; i.e., it is possible to avoid operations with solutions obtained at intermediate steps. The corresponding result was reported in [22] and has been called the Crum's theorem. Let us consider this approach in application to our equation. Let n functions $w_0^{(0)}$, $w_0^{(1)}$, ..., $w_0^{(n-1)}$ describe certain chosen solutions of the initial equation with the potential V_0, these solutions presetting a specific multiple transformation, and let w be one more solution obtained for the same equation so that its choice gives the solution at the nth step. According to the Crum's theorem, the first n solutions provide the following potential after the n-fold Darboux transformation:

$$V_n = V_0 - 2\delta \frac{\partial^2}{\partial x^2} \ln W\left(w_0^{(0)}, \ldots, w_0^{(n-1)}\right). \quad (28)$$

At the same time, the solutions to Eq. (27) are expressed as

$$w_n = \frac{W\left(w_0^{(0)}, \ldots, w_0^{(n-1)}, w\right)}{W\left(w_0^{(0)}, \ldots, w_0^{(n-1)}\right)}, \quad (29)$$

where $W(\ldots)$ are the corresponding Wronskians. For example, in the case of a single transformation, only one function $w_0^{(0)} = v_0 = e^{\delta t}\cosh x$ corresponds to the steady-state solution of the Burgers equation $s_0 = -2\delta \partial \ln v_0/\partial x$ from Eq. (14). Then, $W\left(w_0^{(0)}\right) = v_0$, $W\left(w_0^{(0)}, w\right) = v_0 \partial w/\partial x - w \partial v_0/\partial x$, and formula (29) takes the form of Darboux transformation (8), $w_1 = \left(s_0/(2\delta) + \partial/\partial x\right) w$, as one would expect.

To analyze the sequence of inhomogeneities (25) with the use of Eqs. (28) and (29), it is necessary to determine the explicit form of the functions $w_0^{(k)}(x,t)$. Since the aforementioned inhomogeneities of the Burgers equation are related to the functions V_n through Eq. (5), we can choose V_n to be independent of t. Let v_n represent the solutions of the heat transfer equation with the potential V_n that are used in constructing the next-step potential in the multiple Darboux transformation procedure: $V_{n+1} = V_n - 2\delta \partial^2 \ln v_n / \partial x^2$. Remember that the quantities v_n are related to s_n through the Hopf–Cole transformation. According to Eq. (29), $w_0^{(n)}$ is a solution to the initial equation with zero potential such that the substitution of this solution in Eq. (29) in place of w yields v_n:

$$v_n = \frac{W\left(w_0^{(0)}, \ldots w_0^{(n-1)}, w_0^{(n)}\right)}{W\left(w_0^{(0)}, \ldots w_0^{(n-1)}\right)}. \quad (30)$$

Let us express the solution v_n to the heat transfer equation with the potential

$$V_n = -\delta n(n+1)\cosh^{-2} x, \quad (31)$$

corresponding to $F_n = 2\delta \partial V_n / \partial x$ from Eq. (25). According to the Hopf–Cole transformation, the solution to the Burgers equation $s_n = (n+1)s_0$ corresponds to the solution of the heat transfer equation of the form $v_n = C(t)(\cosh x)^{n+1}$, where C is a certain function of time alone. The explicit form of $C(t)$ is obtained from the condition that v_n is the solution to heat transfer equation (27) with potential (31):

$$v_n(x,t) = e^{\delta(n+1)^2 t}(\cosh x)^{n+1}. \quad (32)$$

Note that, at every step, the operators of the Darboux transformation with the chosen $s_n(x) \sim s_0(x)$ are independent of t. Then, at every step of the Darboux transformation, as the prototype, we can choose a function with an exponential dependence on t of the form of Eq. (32). As a result, the time dependence of $w_0^{(n)}$ should be the same as that of v_n; i.e., $w_0^{(n)}(x,t) =$

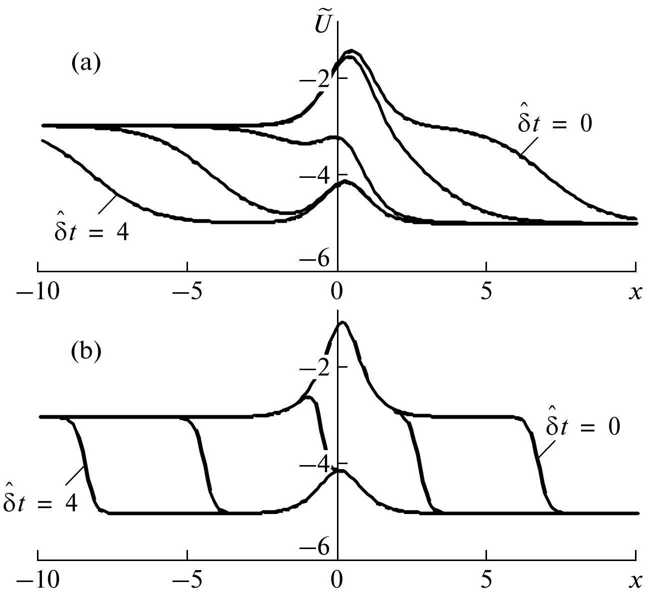

Fig. 4. Passage of the steplike perturbation through the inhomogeneity region for media with different viscosities; the calculations are performed on the basis of Eq. (26). The wave profiles $\tilde{U} = u_n/(2\hat{\delta})$ are shown for successive instants of time $\tilde{d}t = 0, 1, 2, 3, 4$. (a) The solution obtained by a single application of the Darboux transformation. (b) The solution obtained by the eightfold application of the Darboux transformation; in this case, the viscosity coefficient is six times smaller than that in the previous case.

$e^{\delta(n+1)^2 t}\Psi_n(x)$. Substituting this kind of solution in the heat transfer equation, we arrive at the equation $d^2\Psi_n/dx^2 = (n+1)^2\Psi_n$. This yields

$$\Psi_n = A_n\cosh[(n+1)x] + B_n\sinh[(n+1)x], \qquad (33)$$

where A_n and B_n are arbitrary constants. Note that the Darboux transformation $w_n = \hat{D}w_{n-1}$, where $\hat{D} = (s_{n-1}/(2\delta) + \partial/\partial x)$, leads to a change of parity, because both the function $s_{n-1}(x)$ and the differentiation operator $\partial/\partial x$ are odd. According to Eq. (32), the function $v_n(x,t)$ is an even function of coordinate. This means that, if it is the result of an even number of Darboux transformations, the initial function $w_0^{(n)}$ should be even in x; if the number of transformations is odd, the initial function should be odd. In view of all this, one of the coefficients, A_n or B_n, involved in Eq. (33) should be zero. As a result, we obtain the desired expression for the function $w_0^{(n)}$ (without loss of generality, we can take the arbitrary factor to be unity):

$$w_0^{(n)}(x,t) = e^{\delta(n+1)^2 t} \times \begin{cases}\cosh[(n+1)x], & n = 0, 2, 4, \ldots\\ \sinh[(n+1)x], & n = 1, 3, 5, \ldots\end{cases} \qquad (34)$$

We note that the corresponding solutions of the homogeneous Burgers equation have the form of a hyperbolic tangent (for even numbers n) or cotangent (for odd n); i.e., both smooth and singular steady-state profiles are used.

Thus, the solutions to inhomogeneous Burgers equation (20) with right-hand side (25) can be explicitly expressed through the solutions $w(x,t)$ of the heat transfer equation with zero potential, $\partial w/\partial t = 2\delta\partial^2 w/\partial x^2$. Applying the Hopf–Cole transformation to Eq. (29), we express the solution in the form

$$u_n = -2\delta\frac{\partial}{\partial x}\ln\left[\frac{W\left(w_0^{(0)},\ldots w_0^{(n-1)}, w\right)}{W\left(w_0^{(0)},\ldots w_0^{(n-1)}\right)}\right]. \qquad (35)$$

With allowance for Eqs. (5), (25), and (28), formula (35) can be represented in a form that requires a smaller amount of calculation:

$$u_n = 2\delta\left\{\frac{n(n+1)}{2}\tanh x - \frac{\partial}{\partial x}\ln\left[W\left(w_0^{(0)},\ldots w_0^{(n-1)}, w\right)\right]\right\}. \qquad (36)$$

Figure 4b shows an example of the solution constructed with the use of Eq. (36) for $n \gg 1$ ($n = 8$). For comparison, in Fig. 4a we present the solution for the case of a single Darboux transformation. We consider the passage of a steplike perturbation through the inhomogeneity region, by analogy with the case illustrated in Fig. 3b. One can see that, while the solution obtained at $n = 1$ describes jumps with the viscous transition width on the order of the size of the inhomogeneity region, the solution obtained for $n \gg 1$ corresponds to a wave with a much smaller transition width, i.e., in fact, to a shock wave.

DISCUSSION

The approach used in this paper for analyzing the inhomogeneous Burgers equation is based on the combined application of two mathematical methods. The first method is the Hopf–Cole transformation; it relates the solutions of the nonlinear Burgers equation with the right-hand side describing the inhomogeneity to the solutions of the linear heat transfer equation with the potential corresponding to this right-hand side. The second method is the Darboux transformation; it relates the solutions of the heat transfer equations with different potentials. By single or multiple application of the Darboux transformation, from one integrable heat transfer equation (e.g., with zero potential), we obtain a new integrable heat transfer equation with a new potential. By means of the Hopf–Cole transformation, this new integrable heat transfer equation is reduced to the new integrable inhomogeneous Burgers equation.

In this paper, the proposed approach is used to analyze the inhomogeneous Burgers equations the right-hand sides of which describe de Laval nozzles with different contraction ratios. This example illustrates the possibility of obtaining explicit analytical formulas in the case of the n-fold application of the Darboux transformation. We note that, although the corresponding formula (36) seems to be compact, the expansion of the Wronskian involved in it may lead to rather cumbersome expressions. Therefore, direct calculation of the resulting expressions may present considerable difficulties. However, since calculations of determinants and derivatives are routine procedures, the calculations can be performed with the use of standard software for symbolic calculations. In particular, the solution obtained for $n = 8$ and represented in Fig. 4 was calculated using the Maple software package by a PC with a 1.7-GHz processor, the computer time being less than 1 min.

Owing to the arbitrary selection of the function involved in the Darboux transformation and to the possibility of a multiple application of this transformation, the proposed approach can be used for analyzing a wide class of steady-state and nonstationary right-hand sides of the inhomogeneous Burgers equation.

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