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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

Standing Shear Waves in Rubberlike Layered Media

V. G. Andreev, T. B. Krit, and O. A. Sapozhnikov
Faculty of Physics, Moscow State University, Moscow, 119991 Russia
e-mail: timofey@acs366.phys.msu.ru
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Abstract—Standing shear waves arising in layered media the shear modulus of which varies in a stepwise manner at the plain boundaries between the layers are considered. A general solution is obtained for the shear wave amplitudes in a resonator with an N-layer structure the lower boundary of which performs harmonic vibrations while a finite-mass plate is attached to the upper boundary. Results of calculations and measurements are presented for a resonator with a structure in which nondeformable metal layers alternate with elastic rubberlike polymer layers. It is shown that the resonance frequencies of such a resonator can be controlled by changing the number of layers and their thicknesses. It is demonstrated, both experimentally and theoretically, that, from the resonance curve of a resonator with a two-layer structure, it is possible to determine the shear modulus of one of the layers under the condition that the elasticity of the other layer is known. The method of separation into a finite number of layers is used to analyze the resonance characteristics of a one-dimensional resonator filled with a rubberlike medium the properties of which continuously vary in the direction perpendicular to the shear displacements. The choice of the number of layers depending on the type of inhomogeneity is analyzed.

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INTRODUCTION

Interest in studying shear waves in rubberlike inhomogeneous media is primarily caused by the progress in the development of medical diagnostics of soft biological tissues. Pathological changes in soft tissues lead to considerable changes in shear elasticity, and, therefore, visualization of the shear modulus provides the possibility of detecting disease at an early stage [1]. The inhomogeneities of shear elasticity in soft tissues are detected either by measuring the strain inside the tissue under a quasistatic load [2–4] or by measuring the local velocity of shear waves excited by focused ultrasound [5, 6].

The development of adequate models of biological tissues with elastic inhomogeneities for mastering the methods of their diagnostics and mapping is a topical problem. Layered media with plane layer boundaries, where the shear modulus varies in a stepwise manner at the transition from one layer to another, represent simple models of soft biological tissues in which the wave processes are described by one-dimensional equations. In [3], a gelatin layer was used to optimize the profile of the die used in measuring the mechanical impedance of muscle tissue. In [7], it was shown that the brachium muscle is adequately modeled by a three-layer structure. The basic possibility to localize an inhomogeneity of the shear modulus to within fractions of millimeter with the use of shear waves excited by focused ultrasound was demonstrated in [8] by the example of a three-layer medium.

The results obtained for a discrete layered medium can be generalized to the case where the inhomogeneity is continuously distributed along a chosen direction. Gelatin-based hydrogels change their elasticity even in the case of a small pH variation, which is used to create phantoms of soft biological tissues with a smooth inhomogeneity of the shear modulus [9]. Such a medium can be represented by a set of layers with the shear modulus varying in a discrete manner at sufficiently small steps to provide the necessary accuracy in calculating the wave processes. The choice of the necessary number of layers depending on the type of inhomogeneity and the rate of spatial variation of the shear modulus represents a separate problem, which is also analyzed in this paper.

In [10], we studied shear waves propagating in a homogeneous rubberlike layer with a finite-mass plate attached to its upper boundary while its lower boundary performed harmonic vibrations with a preset acceleration. In such a system, resonances occur at frequencies determined by both the plate mass and the viscoelastic parameters of the layer. From the measured resonance curves, the shear modulus and the shear viscosity coefficient of the rubberlike layer were determined. For this purpose, the results of measurements were compared with calculations performed by analytic formulas for a one-dimensional resonator with preset boundary conditions. The values of the shear modulus and the shear viscosity coefficient were determined from the minimal rms deviation of the results of calculations from the measured resonance curve. In the present paper, we generalize the

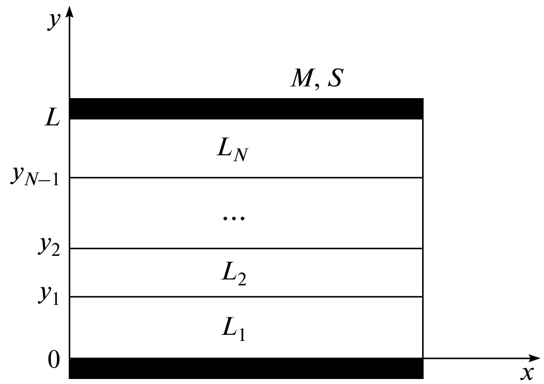

Fig. 1. Model of a multilayer specimen loaded with a finite-mass plate.

approach developed in [10] to the case in which the resonator is formed by a multilayer structure with a plate of a known mass being attached to its upper boundary.

STANDING SHEAR WAVES IN A RESONATOR WITH AN *N*-LAYER STRUCTURE

We consider a structure in the form of a rectangular parallelepiped that has a thickness L and an upper face with an area S and consists of N finite-thickness layers (Fig. 1). The layer of number n is characterized by the thickness L_n, the density ρ_n, the shear modulus μ_n, and the shear viscosity coefficient η_n. We assume that the thickness of the specimen L is much smaller than its transverse size. This assumption allows us to consider the motion of particles as a function of only the longitudinal coordinate, i.e., to use the one-dimensional approximation. The specimen is fixed on a horizontal plate, which performs harmonic vibrations in the direction of the x axis. A plate with the mass M lies on the upper face of the specimen ($y = L$), and the area of the plate is identical to the area of this face.

The elastic processes that occur in each of the layers are described by the equation of motion and by the Hooke's law:

$$\rho_n \frac{\partial^2 u_x^{(n)}}{\partial t^2} = \frac{\partial \sigma_{xy}^{(n)}}{\partial y}, \tag{1}$$

$$\sigma_{xy}^{(n)} = \mu_n \frac{\partial u_x^{(n)}}{\partial y} + \eta_n \frac{\partial^2 u_x^{(n)}}{\partial y \partial t}. \tag{2}$$

Here, $u_x^{(n)}$ is the particle displacement along the x axis; $\sigma_{xy}^{(n)} = F_x^{(n)}/S$ is the stress tensor component; $F_x^{(n)}$ is the force component along the x axis; and $n = 1, 2, \ldots, N$ is the number of a layer.

At the boundaries of neighboring layers with the coordinates $y_n = \sum_{j=1}^{n} L_j$ $(n = 1, 2, \ldots, N-1)$, the continuity conditions for the medium and the equality of mechanical stresses should be satisfied:

$$u_x^{(n)}(y_n, t) = u_x^{(n+1)}(y_n, t), \tag{3}$$

$$\mu_n \frac{\partial u_x^{(n)}}{\partial y}(y_n, t) + \eta_n \frac{\partial^2 u_x^{(n)}}{\partial y \partial t}(y_n, t) = \mu_{n+1} \frac{\partial u_x^{(n+1)}}{\partial y}(y_n, t) + \eta_{n+1} \frac{\partial^2 u_x^{(n+1)}}{\partial y \partial t}(y_n, t). \tag{4}$$

In addition, at the upper and lower boundaries, the two following conditions should be satisfied. The first condition is the preset value of the lower plate acceleration, and the second condition is determined from the law of motion of the upper plate:

$$\left.\frac{\partial^2 u_x^{(1)}}{\partial t^2}\right|_{y=0} = w_x|_{y=0}, \tag{5}$$

$$M \left.\frac{\partial^2 u_x^{(N)}}{\partial t^2}\right|_{y=L} = -\sigma_{xy}^{(N)}\Big|_{y=L} S. \tag{6}$$

For the case of harmonic vibrations $u_x^{(n)}(y,t) = U_n(y)\exp(-i\omega t)$ and $\sigma_{xy}^{(n)}(y,t) = \Sigma_n(y)\exp(-i\omega t)$, Eqs. (1) and (2) take the form

$$-\omega^2 \rho_n U_n = \frac{d\Sigma_n}{dy}, \tag{7}$$

$$\Sigma_n = \tilde{\mu}_n \frac{dU_n}{dy}, \tag{8}$$

where $U_n(y)$ and $\Sigma_n(y)$ are the complex amplitudes of the respective quantities $u_x^{(n)}(y,t)$ and $\sigma_{xy}^{(n)}(y,t)$ and $\tilde{\mu}_n = \mu_n - i\omega\eta_n$ is the complex shear modulus. For the complex amplitude of the transverse particle displacement in the layer, from Eqs. (7) and (8) we obtain the equation

$$\frac{d^2 U_n}{dy^2} + \omega^2 \frac{\rho_n}{\tilde{\mu}_n} U_n = 0. \tag{9}$$

The general solution to Eq. (9) has the form of two opposite waves:

$$U_n = A_n \exp(ik_n y) + B_n \exp(-ik_n y), \tag{10}$$

where the wave number is determined from the dispersion law $k_n = \omega\sqrt{\frac{\rho_n}{\tilde{\mu}_n}}$.

In terms of complex amplitudes, in the case of harmonic vibrations, matching conditions (3) and (4) and boundary conditions (5) and (6) take the form

$$U_n(y_n) = U_{n+1}(y_n), \tag{11}$$

$$\tilde{\mu}_n \frac{dU_n}{dy}(y_n) = \tilde{\mu}_{n+1} \frac{dU_{n+1}}{dy}(y_n), \tag{12}$$

$$U_1\big|_{y=0} = -\frac{W_0}{\omega^2}, \tag{13}$$

$$\left(U_N - \frac{\tilde{\mu}_N S}{M\omega^2}\frac{dU_N}{dy}\right)\bigg|_{y=L} = 0. \tag{14}$$

Substituting the solution in the form of Eq. (10) in Eqs. (11)–(14), we arrive at the system of $2N$ linear algebraic equations

$$\begin{cases} A_1 + B_1 = -\dfrac{W_0}{\omega^2}; \\ A_n\exp(ik_n y_n) + B_n\exp(-ik_n y_n) \\ - A_{n+1}\exp(ik_{n+1}y_n) - B_{n+1}\exp(-ik_{n+1}y_n) = 0; \\ A_n ik_n\tilde{\mu}_n\exp(ik_n y_n) - B_n ik_n\tilde{\mu}_n\exp(-ik_n y_n) \\ - A_{n+1}ik_{n+1}\tilde{\mu}_{n+1}\exp(ik_{n+1}y_n) \\ + B_{n+1}ik_{n+1}\tilde{\mu}_{n+1}\exp(-ik_{n+1}y_n) = 0; \\ n = 1,2\ldots,N-1; \\ A_N\left(1 - ik_N\dfrac{\tilde{\mu}_N S}{M\omega^2}\right)\exp(ik_N L) \\ + B_N\left(1 + ik_N\dfrac{\tilde{\mu}_N S}{M\omega^2}\right)\exp(-ik_N L) = 0. \end{cases} \tag{15}$$

System (15) was solved according to Cramer's formulas [11] using MatLab. We calculated the $2N$ amplitudes of the opposite waves and the ratio of the accelerations of the upper and lower plates (W_L/W_0). Then, this ratio was measured experimentally.

Calculations showed that vibrations of the multilayer structure under study are of resonance nature. The resonance frequencies at which the ratio W_L/W_0 reaches maximal values are determined by both the size of the specimen and the viscoelastic parameters of the layers.

A RESONATOR WITH AN 11-LAYER STRUCTURE

As an example, we consider a specimen in which soft and rigid layers alternate. The layers with odd numbers are made from a rubberlike material the shear modulus of which is several orders of magnitude smaller than the bulk modulus. The layers with even numbers are made from ordinary hard material in which the shear and bulk elasticities are of the same order of magnitude. When this specimen performs shear vibrations, only the soft layers are deformed, whereas the hard layers are displaced as a whole without deformation. In such a structure, it is possible to obtain considerable shear strains in the rubberlike layers at low frequencies. Maximal shear displacements and strains are achieved in these layers in the case of vibrations at the first resonance frequency. In this case, the smaller the layer thickness is at a fixed ratio W_L/W_0, the greater the strain. In a homogeneous layer with a thickness L, the first resonance frequency increases with decreasing thickness, which does not allow obtaining considerable strains in thin homogeneous layers. In a multilayer structure with hard layers, the latter play the role of a distributed mass and reduce the first resonance frequency even for small values of soft layer thickness. This makes it possible to create structures with controlled elastic parameters, both linear and nonlinear ones, and observe nonlinear wave effects in small volumes. A detailed review of nonlinear effects in resonators under the conditions of shock front formation can be found in [12].

Calculations and measurements were carried out for a resonator with an 11-layer structure, the thickness of which was $L = 32$ mm and transverse dimensions were 7×4 cm; the structure was loaded with a plate with the mass $M = 10.87$ g. Six soft layers with a thickness of 2 mm each were made from a polymer material called plastisol (by MF Manufacturing, United States). The soft layers were in contact with the lower and upper plates. According to static measurements by pressing a hard ball into a homogeneous plastisol layer, the shear modulus of this material was found to be 24 ± 3 kPa. Five hard layers lying between the plastisol ones were duralumin plates with a thickness of 4 mm each.

The measurements were performed using the experimental setup that was described in detail in [10]. The resonator was fixed to the lower plate, which was firmly connected with a vibrator (type 4810 by Bruel&Kjaer, Denmark). A sine voltage was supplied to this vibrator from the signal generator (AFG 3021B, Tektronix, United States) through a low-frequency amplifier (LV 103, Robotron, Germany). Two small-size accelerometers (type 4374, Bruel&Kjaer) were positioned on the upper and lower plates to measure their accelerations. The signals from the accelerometers were sent through charge amplifiers (type 2635, Bruel&Kjaer) to the inputs of a two-channel digital oscilloscope (TDS 3032B, Tektronix). The digitized signals from the oscilloscope were supplied through a GPIB interface to a computer where the ratio of the acceleration amplitudes was calculated. The measurements were performed at a fixed acceleration amplitude of the lower plate. The frequency was varied at a step of 0.1 Hz. With each of the changes in frequency, the voltage amplitude at the vibrator was adjusted so as to provide the preset value of the acceleration amplitude of the lower plate.

Figure 2 shows the results of measurements for the resonator with the 11-layer structure by dots. The first three resonances are observed at frequencies of 44.5, 140, and 230 Hz. The resonance peak heights decrease and the resonance curve widths increase with increas-

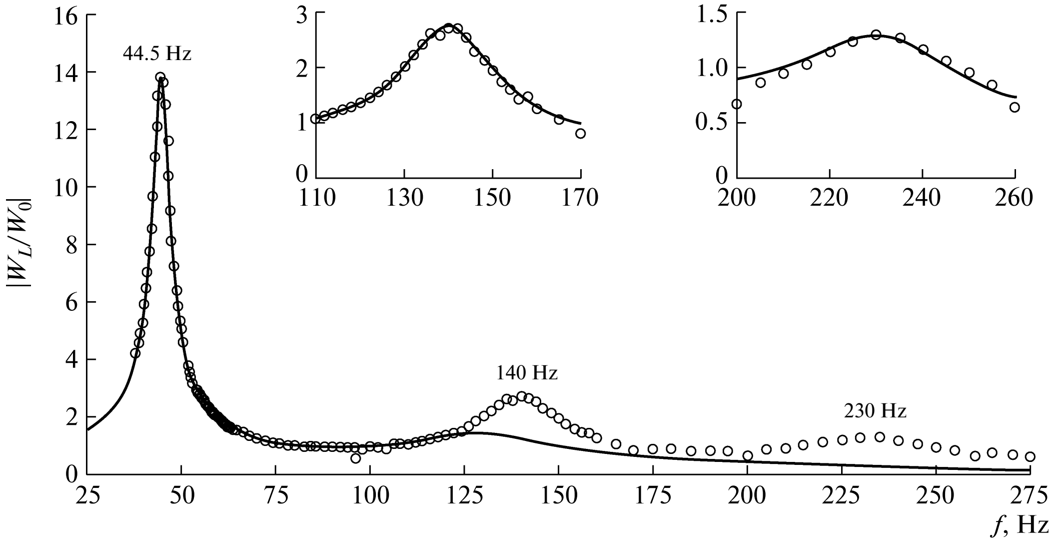

Fig. 2. Ratio of acceleration amplitudes of the upper and lower plates in a resonator with an 11-layer structure (6 plastisol layers with a thickness of 2 mm each and 5 duralumin layers with a thickness of 4 mm each). The dots represent the results of measurements, and the line is the result of calculations for the plastisol parameters $\mu = 25.3$ kPa and $\eta = 8.4$ Pa s. The insets represent the results of measurements and calculations near the second ($\mu = 29$ kPa, $\eta = 4.8$ Pa s) and third ($\mu = 30$ kPa, $\eta = 3.5$ Pa s) resonance frequencies.

ing number of resonance. The values of the shear modulus and the shear viscosity of plastisol were chosen in such a way that the calculated resonance curve near the first resonance frequency was as close as possible to the results of measurements. The shear modulus of duralumin was taken to be 42×10^6 kPa, and the shear viscosity of this material was taken to be zero. In Fig. 2, the calculated curve is shown by the solid line. As a result, at the frequencies near the first resonance, we found the following values of the shear modulus and the shear viscosity coefficient of plastisol: $\mu = 25.3 \pm 0.8$ kPa and $\eta = 8.4 \pm 0.4$ Pa s. As it was noted in [10], the viscoelastic parameters of plastisol depend on frequency. Therefore, at frequencies above 125 Hz, the results of measurements deviate from the curve calculated for viscoelastic parameters at the first resonance frequency (44.5 Hz). At frequencies above 125 Hz, the viscoelastic parameters were determined from the coincidence of the calculated and experimental curves near the second and third resonances (the insets in Fig. 2). This provided the following values of the shear modulus and the shear viscosity coefficient: $\mu = 29 \pm 1$ kPa and $\eta = 4.8 \pm 0.3$ Pa s at the second resonance frequency (140 Hz) and $\mu = 30 \pm 1$ kPa and $\eta = 3.5 \pm 0.4$ Pa s at the third resonance frequency (230 Hz).

To determine the shear strains in different plastisol layers, we calculated the amplitude profile of the transverse particle displacements in the standing wave, $U(y)$, at the first resonance frequency. Figure 3a shows the wave profile at a frequency of 44.5 Hz and at the lower plate acceleration amplitude $W_0 = 5$ m/s^2 for the instant of time corresponding to the maximal displacement of the upper resonator boundary. At the first resonance frequency, slightly less than a quarter of the shear wavelength fits within the thickness L of the structure. The sloping parts of the profile correspond to the particle displacements in the soft plastisol layers. The duralumin plates are displaced as a whole, without any deformations, which is illustrated by the vertical segments in the wave profile. Since the wave profile varies with time, the maximal strain is only observed at the instants corresponding to the maximal displacement of the upper resonator boundary from the equilibrium position. For the profile shown in Fig. 3a, the maximal strain in each of the layers was calculated by the formula $\varepsilon_{max} = \Delta U_n / L_{pl}$, where ΔU_n is the difference between the displacement amplitudes of the upper and lower boundaries of the layer and $L_{pl} = 2$ mm is the plastisol layer thickness. Figure 3b shows the distribution of the shear strain throughout the plastisol layers. The strain is maximal in the lower layers and decreases toward the upper boundary. In the lowest plastisol layer ($n = 1$), the strain value is 0.052, which is more than four times greater than the strain averaged over the whole specimen thickness.

In a resonator with a multilayer structure, the first resonance frequency depends on the number of layers. As the number of layers increases, the resonance frequency decreases while the acceleration amplification coefficient increases. For example, for a structure consisting of plastisol and duralumin layers, the resonance frequency varies from 65 (for $N = 7$) to 27 Hz (for $N = 19$). By taking materials with different densities for the hard layers, it is also possible to change the

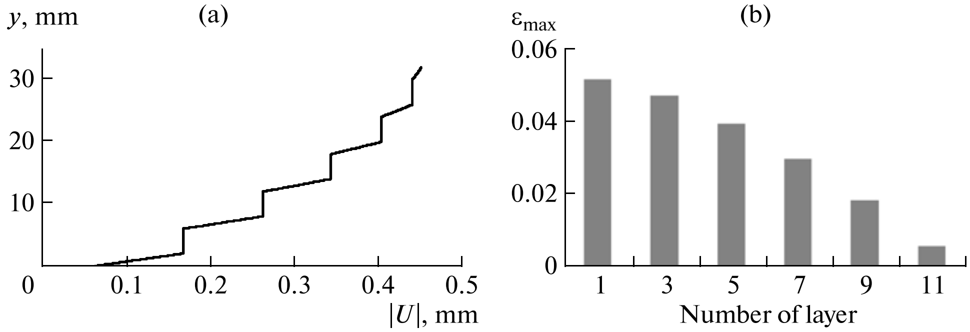

Fig. 3. (a) Profile of the particle displacement amplitude in the standing wave at the first resonance frequency (44.5 Hz) for the lower plate acceleration amplitude $W_0 = 5$ m/s^2 at the instant corresponding to the maximal displacement of the upper boundary of the resonator. The sloping and vertical lines show the displacements in the plastisol and duralumin layers. (b) Strain distribution over the plastisol layers.

resonance frequency and the amplification in a multilayer resonator.

MEASUREMENT OF THE SHEAR MODULUS OF ONE OF THE LAYERS IN A TWO-LAYER STRUCTURE

A two-layer structure represents a simple model of an inhomogeneous medium. The shear modulus of one of the layers can be determined from the measured resonance curves of a two-layer resonator under the condition that the shear modulus of the other layer is known.

Calculations were performed for a resonator with a structure in which the shear modulus of the hard layer was known to be 6.0 kPa. The shear modulus of the second layer was approximately half that of the first layer, but its exact value was unknown. The shear viscosity coefficient was the same for both layers and came to 3.5 Pa s. To indicate the parameters of the layers with the known and unknown shear moduli, we use the subscripts 0 and x, respectively. The two-layer structure had the form of a rectangular parallelepiped with sides of 39 and 70 mm and a thickness of 26 mm. Calculating the resonance curves, we assumed that a solid plate with a mass of 3 g lay on the upper boundary of the structure, which corresponded to experimental conditions.

Figure 4 shows the dependences of the first resonance frequency of the two-layer specimen on the shear modulus μ_x. The plots in Figs. 4a and 4b correspond to the cases in which the layer with a known modulus of 6.0 kPa lies below the other layer and above it, respectively. The thick black lines represent the dependences of the first resonance frequency, which were calculated under the condition that the thicknesses of the two layers are identical. It should be noted that, in the case in which the harder layer lies below the softer layer (Fig. 4a), the values of the resonance frequencies are higher than those in the case in which the hard layer is the upper one (Fig. 4b). From the values of the resonance frequencies measured in the first and second cases (represented by dots in Fig. 4), we can determine the values of the shear modulus of the softer layer. It was found to be 3.5 kPa, which coincided with the value obtained from the static measurements with a hard ball pressed into the material. The error in measuring the resonance frequency is shown by two thin horizontal lines in Fig. 4; in our case, it was 1%. This error leads to a corresponding error in determining the shear modulus. Since, in Fig. 4b, the value of the derivative $df_1/d\mu_x$ is higher than the corresponding value in Fig. 4a, the aforementioned algorithm of shear modulus determination gives a smaller error in the case in which the softer layer with the unknown modulus lies below the hard layer. From the results of measurements, we obtained $\mu_x = 3.5 \pm 0.3$ kPa for the softer layer being the upper one and $\mu_x = 3.5 \pm 0.1$ kPa for the softer layer being the lower one.

In practice, the thickness of an individual layer can be determined with a certain error. This is due to the technology used for fabrication of layered specimens, the diffusion of the components responsible for the polymer stiffness, and the roughness of the layer boundary. At the same time, the required thickness of the entire specimen can be maintained with a good accuracy. Therefore, we estimated the error in determining the shear modulus in the presence of a 5% error for the thickness of each of the layers. In Fig. 4, the gray dashed lines show the dependences of the first resonance frequency on the shear modulus of the soft layer, which were calculated for the case in which the upper layer thickness is 13.65 mm and the lower layer thickness is 12.35 mm. The solid grey lines represent the dependences calculated for the case in which the upper layer has a thickness of 12.35 mm and the lower layer has a thickness of 13.65 mm. Thus, an error of ±5% in the layer thickness leads to doubling of the error in determining the shear modulus.

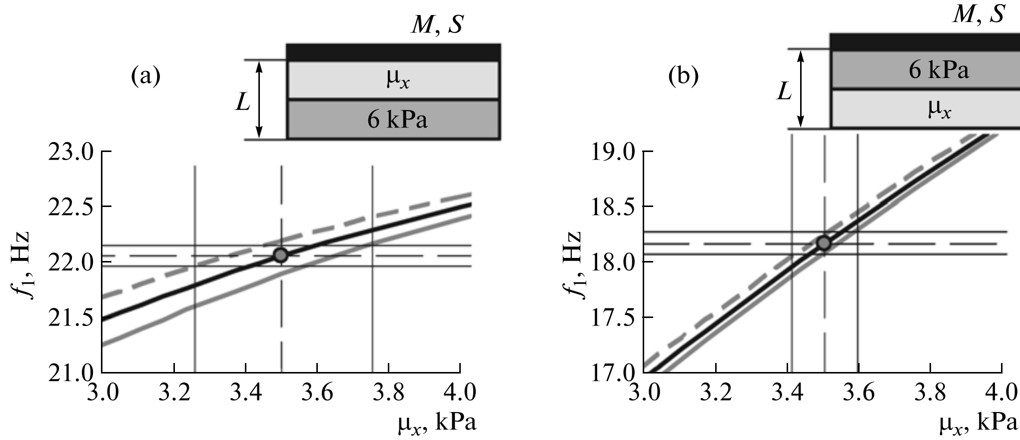

Fig. 4. Calculated dependences of the first resonance frequency on the shear modulus of the soft layer for the cases where the soft layer lies (a) above and (b) below the other layer. The black line shows the results of calculation for identical layer thicknesses of 13 mm. The grey lines represent similar dependences for an upper layer thickness of 13.65 mm and a lower layer thickness of 12.35 mm (the dashed grey lines) and for an upper layer thickness of 12.35 mm and a lower layer thickness of 13.65 mm (the solid grey lines). The measured frequency value is indicated by the thick dot.

USE OF THE LAYERED MEDIUM MODEL IN CALCULATIONS FOR RESONATORS WITH AN INHOMOGENEOUS RUBBERLIKE MEDIUM

The layered medium model is convenient for studying the media the shear modulus of which varies in a known manner along only one coordinate axis.

We consider two specimens with a thickness of 10 mm and assume that, in these specimens, the shear modulus depends on depth according to the quadratic law with the minimum at the half-thickness (Fig. 5). At the surface of both of these specimens, the shear modulus is 6 kPa. The minimal values of the shear modulus are 4.5 and 1.0 kPa in the first and second specimens, respectively. Both specimens are shaped as a parallelepiped with its sides being 40 and 70 mm.

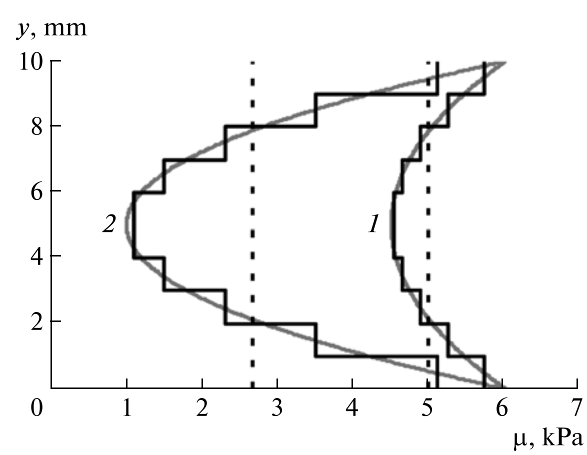

Fig. 5. Distribution of the shear modulus in thickness for specimen nos. 1 and 2 (the grey lines). Division of the specimens into ten identical-thickness layers with a constant value of the shear modulus within each of the layers (the black lines). The dashed lines indicate the values of the effective shear modulus μ_{eff}.

To calculate the resonance curves of resonators with the aforementioned specimens, we divided the latter into ten layers with identical thicknesses of 1 mm and with a constant value of the shear modulus within each of the layers. The value of the shear modulus in a given layer was determined from the initial distribution by taking the value corresponding to the middle of this layer. The resulting stepped distributions are shown in Fig. 5 by black solid lines. Near the vertices of parabolas in the two central layers, the same values of the shear modulus were preset. Table 1 shows the values of the shear modulus for individual layers of the two specimens. The shear viscosity coefficient was assumed to be the same for both specimens: 2 Pa s. We calculated the shear waves in the resonator with inhomogeneous specimens with upper boundaries loaded with a solid plate with a mass of 8 g.

Table 2 shows the results of calculating the first resonance frequency for the resonators with the two specimens. The frequency f_{1L} was obtained with allowance for the division of the specimens into ten layers each. For comparison, we present the frequency f_{10} obtained for the division into 150 layers, in which case the distribution of the shear modulus in thickness can be considered as continuous. This value was assumed to be exact and corresponding to the parabolic profile of the inhomogeneity. For the first specimen, where the inhomogeneity is weaker in comparison with the second specimen, the division into ten layers gives a result that coincides with the exact value. The corresponding resonance curves also coincide to a high accuracy. For the resonator with the second specimen where the inhomogeneity is more pronounced, the frequency values differ by 0.5 Hz, which does not exceed 2%. An increase in the number of layers to 20 leads to an exact

Table 1. Values of the shear moduli (in kPa) for the layers of two specimens (nos. 1 and 2)

Specimen no. \ Number of layer	1	2	3	4	5	6	7	8	9	10
1	5.73	5.25	4.89	4.65	4.53	4.53	4.65	4.89	5.25	5.73
2	5.1	3.5	2.3	1.5	1.1	1.1	1.5	2.3	3.5	5.1

result. It should be noted that, in the case of weak inhomogeneity (specimen no. 1), the homogeneous specimen approximation with the effective modulus given by the formula

$$\mu_{\text{eff}} = \frac{1}{L}\int_0^L \mu(y)dy \tag{16}$$

leads to a result close to the exact one. However, for the strongly inhomogeneous specimen, this approximation considerably overestimates the result. That is, the effective elasticity of the specimen that is calculated by Eq. (16) is higher than the actual elasticity.

CONCLUSIONS

Layered and rubberlike media are of interest not only as objects of investigation, but also as models for studying wave processes in complex inhomogeneous media. In this paper, we described the algorithm of solving the direct problem, where the known viscoelastic parameters of layers are used to determine the amplitudes of standing waves arising in a one-dimensional layered structure under the effect of a harmonic source. We showed that a structure consisting of a set of layers with different elasticities possesses resonance properties, which are governed by both the geometry of the layer arrangement and the viscoelastic parameters of the layers.

The inverse problem is important for diagnostic purposes. This problem was solved for the simple case of a two-layer structure. It was shown that the value of the first resonance frequency can be used for a unique determination of the shear modulus of one of the layers under the condition that the elasticity of the other layer is known. The uncertainty in the values of the layer thicknesses leads to an additional error in the determination of the shear modulus by the proposed method. An analysis of the error was performed for the case in which the layer thickness is known accurate to within ±5%. For a two-layer structure, two measurements of the resonance frequency are possible: first, when the layer with the modulus to be determined is below the other layer and, second, when it is above the other layer. It was found that the error in the determination of the shear modulus is smaller when the layer with the unknown elasticity lies below the layer with the higher elastic modulus.

By choosing the layers with different shear elasticities, it is possible to control both the resonance characteristics and the mechanical parameters of the layered structure. In other words, it is possible to create materials with preset viscoelastic characteristics. Considering the structure with solid duralumin plates alternating with thin rubberlike polymer layers as an example, we showed that local shear strains that are several times greater than the values averaged over the structure arise in some of the polymer layers. In such layers, nonlinear properties of the rubberlike material can manifest themselves at relatively small vibration amplitudes of the whole structure. Specifically, the plastisol used in our experiments exhibits an increase in shear elasticity with increasing strain [13]. This effect becomes noticeable when the strain exceeds a level of 20%, which can be reached in individual layers even at relatively small vibration excitation amplitudes. Thus, we obtain a structure in which the elasticity is a nonlinear function of strain. At the same time, localization of the region with increased elasticity depends on the frequency of vibrations. Such models with elasticity increasing with strain according to a preset law are of practical interest from the point of view of constructing both phantoms of muscle tissue [14] and its theoretical models [15].

The results obtained for standing waves in discrete multilayer structures were generalized to the case of media with a continuous distribution of the shear modulus. For this purpose, we used the approach based on the replacement of the continuous distribution by a set of layers with constant shear moduli. The use of a small number of such layers (30–40) makes it possible to obtain sufficiently accurate solutions for standing waves in media with pronounced inhomogeneities.

Table 2. Values of the first resonance frequency for specimen nos. 1 and 2. The frequency f_{10} corresponds to the exact solution, f_{1L} is calculated for ten layers, and $f_{1\text{eff}}$ corresponds to homogeneous specimens with the effective shear modulus μ_{eff}

Specimen no.	f_{10}, Hz	f_{1L}, Hz	$f_{1\text{eff}}$, Hz
1	43.6	43.6	43.8
2	26.8	27.3	32

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