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Standing Waves in an Elastic Layer Loaded with a Finite Mass

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Abstract—Standing shear waves in a plane-parallel rubberlike layer fixed without slippage between two rigid plates with finite masses are investigated. The lower plate, which underlies the layer, oscillates in the direction parallel to its surface under an external harmonic force, whereas the upper plate freely overlies the layer. It is shown both theoretically and experimentally that such a system exhibits resonances at frequencies the values of which depend on the mass of the free plate and the shear modulus of the layer. The shapes of the resonance curves are calculated and measured for different values of parameters of the layer and different masses of the upper plate. From the measured resonance curves, it is possible to determine the dynamic shear modulus and the shear viscosity of the rubberlike material.

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INTRODUCTION

Rubberlike media (also called soft solids), which include rubbers, gel-type materials, and soft biological tissues, are characterized by the smallness of the shear elastic modulus in comparison with the bulk modulus. In such media, Poisson's ratio is close to 0.5, while the shear wave velocity is much smaller than the velocity of bulk waves and may be as small as several meters per second.

Among the various methods used for studying viscoelastic properties of media, we chose the convenient method based on the analysis of standing waves in a layer of the medium of interest. This approach proved to be effective in the case of longitudinal waves in the megahertz frequency range. The corresponding ultrasonic interferometers are widely used for measuring the velocity and attenuation coefficient of sound in liquids [1]. The attenuation coefficient is determined from the resonance peak width, and the velocity of sound can be found from the values of the resonance frequencies.

A similar approach is suitable for measuring the complex shear modulus of a rubberlike medium [2, 3]. Since the shear wave velocities in rubberlike media are small, the corresponding wavelengths are also small, which makes it possible to excite standing waves in small-size samples at relatively low sonic frequencies. Such measurements have become topical in application to soft biological tissues, the shear modulus of which is an important diagnostic parameter of the tissue state [4–6].

A shear-wave interferometer may be convenient not only for studying the linear elastic parameters of a material, but also for measuring its nonlinear moduli. For example, the presence of a cubic nonlinearity in a rubberlike material gives rise to a number of new nonlinear phenomena, including the resonance frequency shift at finite shear wave amplitudes [7], which allows one to estimate the nonlinear elastic modulus of the material.

In this paper, we present the results of theoretical and experimental studies of standing shear waves in a resonator filled with a rubberlike medium. We investigate the possibilities of using the interferometric method for measuring the linear viscoelastic parameters of rubberlike materials.

STANDING SHEAR WAVES IN A ONE-DIMENSIONAL RESONATOR

The excitation of standing shear waves is illustrated in the inset in Fig. 1. The layer of a rubberlike material with a thickness L is fixed to the lower plate ($x = 0$) so that the motion of the plate is not accompanied by layer slippage. The upper surface of the layer ($x = L$) underlies a finite-mass plate, which moves together with the layer. The plate underlying the layer oscillates in the transverse direction (along the y axis) with the acceleration

$$w_y(x = 0, t) = W_0 \cos\omega t = \mathrm{Re}\left(W_0 e^{-i\omega t}\right).$$

Oscillations of the lower plate cause a shear wave in the layer of the rubberlike material. This wave reaches the upper boundary of the layer, then is reflected from it, and finally is combined with the incident wave, which results in the formation of a standing wave in the layer. We note that elastic oscillations in mass-loaded rubberlike layers can be used for vibration damping in rods and plates [8].

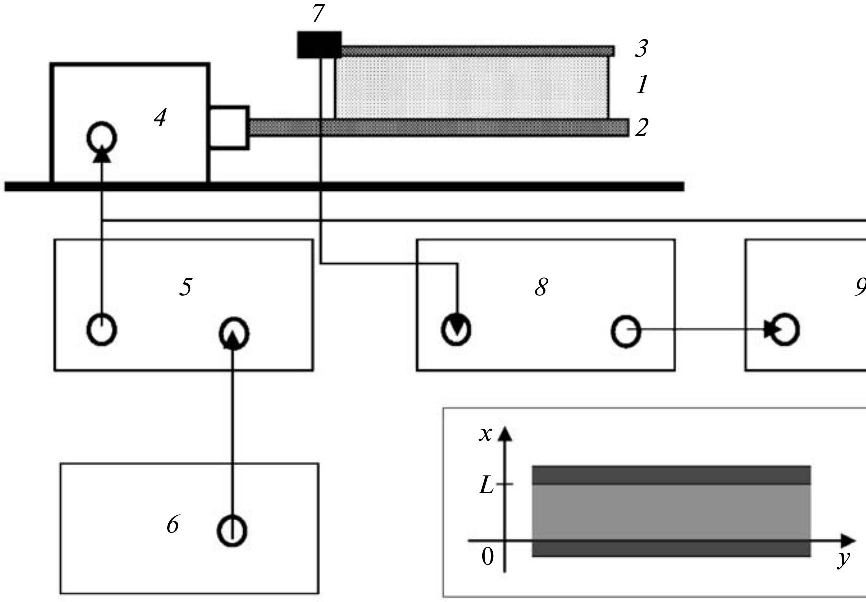

Fig. 1. Schematic representation of the experimental setup: (*1*) layer of the rubberlike material, (*2*) lower plate, (*3*) upper plate, (*4*) vibrator, (*5*) low-frequency amplifier, (*6*) signal generator, (*7*) accelerometer, (*8*) charge amplifier, and (*9*) digital oscilloscope. The inset illustrates the excitation of a standing shear wave in an elastic layer with a width L.

The particle motion in a rubberlike material is described by the equation of motion and Hooke's law [9]:

$$\rho\frac{\partial^2 u_y}{\partial t^2} = \frac{\partial\sigma_{xy}}{\partial x},$$

$$\sigma_{xy} = \mu\frac{\partial u_y}{\partial x} + \eta\frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial t}\right).$$

Here, ρ is the density of the material, u_y is the displacement along the y axis, μ is the shear modulus, η is the shear viscosity coefficient, $\sigma_{xy} = F_y/S$ is the stress tensor component, F_y is the force component along the y axis, and S is the area of the layer surface perpendicular to the x axis.

For harmonic oscillations described by the formulas $u_y(x, t) = U(x)\cdot e^{-i\omega t}$ and $\sigma_{xy}(x,t) = \Sigma(x)\cdot e^{-i\omega t}$, the aforementioned equations have the form

$$-\omega^2\rho\cdot U = \frac{d\Sigma}{dx}, \quad (1)$$

$$\Sigma = \tilde{\mu}\frac{dU}{dx}. \quad (2)$$

Here, $U(x)$ and $\Sigma(x)$ are the complex amplitudes of the respective quantities $u_y(x,t)$ and $\sigma_{xy}(x,t)$ and $\tilde{\mu} = \mu - i\omega\eta$ is the complex shear modulus. For convenience, we introduce the loss angle $\delta = \delta(\omega)$ characterizing the relative role of viscous stresses: $\tan\delta = \omega\eta/\mu$. Then, $\tilde{\mu} = \mu(1 - i\tan\delta) = \mu e^{-i\delta}/\cos\delta$. From equation of motion (1) and Hooke's law (2), for the complex amplitude of the transverse displacement, we obtain the standard wave equation

$$\frac{d^2U}{dx^2} + \frac{\omega^2\rho}{\tilde{\mu}}U = 0. \quad (3)$$

The general solution to this equation has the form of a superposition of two counter-propagating waves:

$$U = Ae^{ikx} + Be^{-ikx},$$

where the wave number is determined from the dispersion law

$$k = \frac{\omega}{\sqrt{\tilde{\mu}/\rho}} = \frac{\omega}{c_t} + i\alpha.$$

Here, we introduced the following notations for the velocity and attenuation coefficient of shear waves:

$$c_t = \frac{\sqrt{\mu/\rho}}{\cos(\delta/2)\sqrt{\cos\delta}}, \quad \alpha = \frac{\omega}{c_t}\tan(\delta/2).$$

The amplitudes of the counter-propagating waves are determined from two boundary conditions. The first condition is the preset value of acceleration of the lower plate, and the second condition is determined from the law of motion of the upper plate:

$$\left.\frac{\partial^2 u_y}{\partial t^2}\right|_{x=0} = w_y|_{x=0}, \quad M\left.\frac{\partial^2 u_y}{\partial t^2}\right|_{x=L} = -\sigma_{xy}|_{x=L}\cdot S,$$

where M is the mass of the upper plate.

For complex amplitudes, the boundary conditions have the form

$$U\big|_{x=0} = -\frac{W_0}{\omega^2}, \quad \left(U - \frac{\mu S}{M\omega^2 \cos\delta} \cdot e^{-i\delta}\frac{dU}{dx}\right)\bigg|_{x=L} = 0, \quad (4)$$

where W_0 is the acceleration amplitude of the lower plate. Solving Eq. (3) with boundary conditions (4) and taking into account that the acceleration amplitude of the upper plate is $W_L = -\omega^2 U\big|_L$, we obtain the following expression for the ratio of the aforementioned complex amplitudes:

$$\frac{W_L}{W_0} = \frac{2i\frac{S}{M\omega}\sqrt{\frac{\rho\mu}{\cos\delta}} \cdot e^{-i\frac{\delta}{2}}}{e^{-ikL}\left(1 + i\frac{S}{M\omega}\sqrt{\frac{\rho\mu}{\cos\delta}} \cdot e^{-i\frac{\delta}{2}}\right) - e^{ikL}\left(1 - i\frac{S}{M\omega}\sqrt{\frac{\rho\mu}{\cos\delta}} \cdot e^{-i\frac{\delta}{2}}\right)}. \quad (5)$$

If the shear viscosity is absent, solution (5) is considerably simplified:

$$\frac{W_L}{W_0}\bigg|_{\delta=0} = \frac{2i\tan\gamma}{e^{-ikL}(1 + i\tan\gamma) - e^{ikL}(1 - i\tan\gamma)} = \frac{\sin\gamma}{\sin(kL - \gamma)}, \quad (6)$$

where $\gamma \equiv \arctan\left(\frac{\rho c_t S}{M\omega}\right)$.

Figure 2 shows the absolute value of the ratio of complex acceleration amplitudes that was calculated by Eq. (5) for different values of the shear modulus and shear viscosity coefficient of the material in the frequency range from 1 to 100 Hz. In these calculations, we used the following basic parameters: the density of the material $\rho = 1$ g/cm^3, the layer thickness $L = 3.2$ cm, the mass of the upper plate $M = 28.3$ g, and its area $S = 94$ cm^2. The solution obtained for the layer with the parameters $\mu = 5$ kPa and $\eta = 2$ Pa s is shown by the thick line in Fig. 2. Resonances are observed at the frequencies 15.97, 47.98, and 79.32 Hz. The thin line shows the solution obtained for $\mu = 10.5$ kPa and $\eta = 4$ Pa s. In this case, the frequencies 23.14 and 69.33 Hz correspond to the first and second resonances. One can see that higher-order resonances are broader and smaller in amplitude; therefore, resonances that occur at frequencies higher than 100 Hz are undetectable at the aforementioned parameters. This determined the choice of the frequency range under study.

From Eq. (6) it follows that, when the mass of the upper plate is infinitely small and the attenuation is absent, resonances occur at the frequencies

$$f_n = \frac{c_t}{4L}(2n-1), \quad n = 1, 2, \ldots \quad (7)$$

We note that the resonance frequency ratios observed in the layer for the first and second sets of parameters are as follows: 15.97 : 46.98 : 79.32 = 1 : 3 : 4.97 and 23.14 : 69.33 = 1 : 3, which agrees well with Eq. (7). This means that, for a layer loaded with a small but finite mass and characterized by a shear viscosity coefficient not exceeding 4 Pa s, the resonance frequencies can be estimated by simple formula (7).

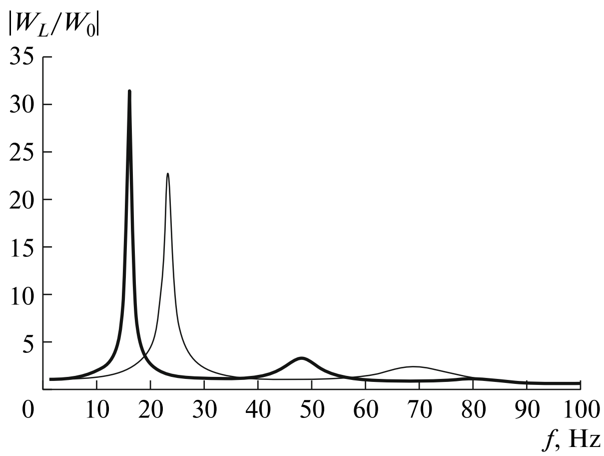

Fig. 2. Acceleration amplitude ratio calculated for an elastic layer with the following parameters: $\mu = 5$ kPa and $\eta = 2$ Pa s (the thick line); $\mu = 10.5$ kPa and $\eta = 4$ Pa s (the thin line).

EXPERIMENTAL RESULTS

Figure 1 schematically illustrates the experimental setup used for studying standing shear waves in a rubberlike material. The layer had the form of a rectangular parallelepiped, the sides of which were 7.9 and 11.9 cm and the thickness was $L = 3.2$ cm; it was made from a soft polymer material called plastisol (produced by MF Manufacturing, United States). The static shear modulus, which was measured by pressing a hard ball into the material [9], was found to be 11.7 kPa. The layer was fixed on a horizontal metal plate, and a thin fabric-based laminate plate with a mass of 28.3 g was placed on the upper surface of the layer. The latter plate oscillated together with the layer surface without slippage. The area of this plate was identical to that of the upper face of the sample. The lower plate was rigidly fixed to a TIRA TV 52120 vibrator. The vibrator was driven by a sine voltage supplied to it from an HP33120A generator through a TIRA BAA 500

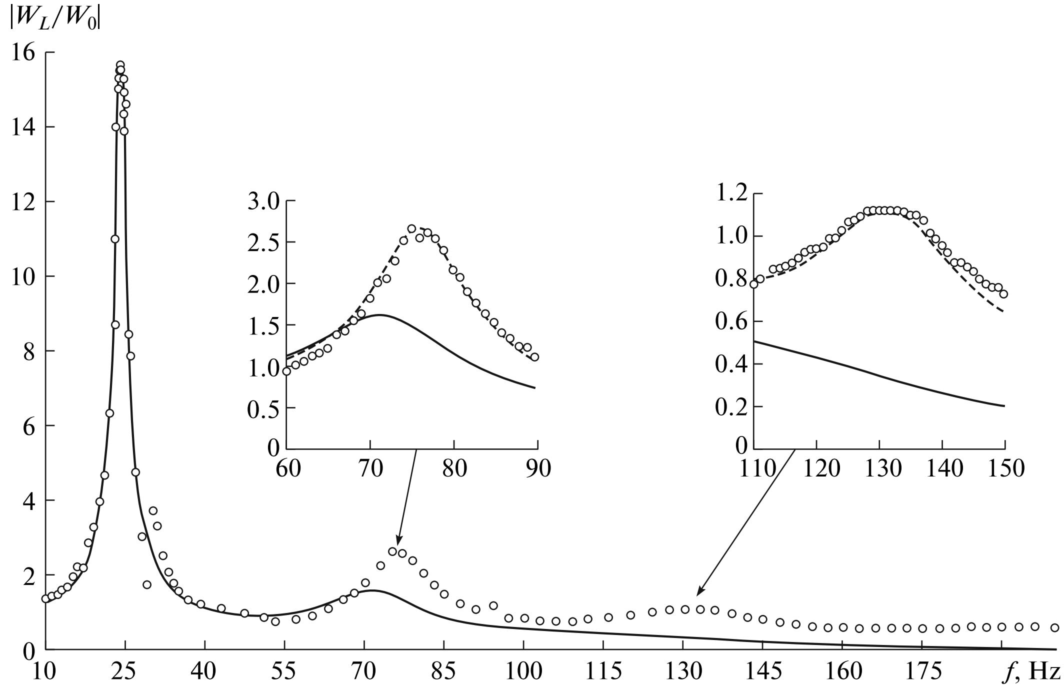

Fig. 3. Experimental frequency dependence of the ratio of acceleration amplitudes of the upper and lower plates (dots) and the corresponding theoretical curve calculated for a layer with the parameters $\mu = 11.3$ kPa and $\eta = 6.2$ Pa s (the solid line). The insets show the parts of the dependence in the frequency bands 60–90 and 110–150 Hz. The dashed curves are obtained by theoretical calculation for a layer with $\mu = 12.6$ kPa and $\eta = 3.9$ Pa s in the frequency band 60–90 Hz and for a layer with $\mu = 13.5$ kPa and $\eta = 2.9$ Pa s in the frequency band 110–150 Hz.

amplifier. Measurements were performed at a fixed amplitude of the generator output voltage for different frequencies, which were varied within 1–200 Hz at a step of 0.1 Hz. Two 4374 B&K accelerometers were positioned on the upper and lower plates to measure the accelerations of the plates. The signals from the accelerometers were supplied via 2635 B&K charge amplifiers to the inputs of a dual-channel Tektronix TDS 3032B digital oscilloscope. The digitized signals from the oscilloscope were supplied through a GPIB interface to a computer, where the acceleration amplitude ratio was calculated.

Figure 3 shows the frequency dependence of the ratio of acceleration amplitudes of the upper and lower plates (dots). The first three resonances of the layer are observed at frequencies of 24, 76, and 131 Hz. The acceleration gain at the first resonance frequency is 16, while the width of the resonance curve at a level of 0.7 is 2 Hz, which corresponds to a quality factor of 12. The acceleration gain at the second resonance frequency of 76 Hz is 3, and the resonance curve width is 11.4 Hz, which corresponds to a quality factor of 6.7. The solid line in Fig. 3 shows the theoretical curve calculated for the layer with $\mu = 11.3$ kPa and $\eta = 6.2$ Pa s. It is in good agreement with the experiment at the frequencies near the first resonance, whereas, at other resonance frequencies, considerable discrepancies are observed.

To provide a coincidence of experimental and theoretical values near the second resonance in the frequency band 60–90 Hz, the preset value of the shear modulus was increased to 12.6 kPa and the preset value of the shear viscosity coefficient was reduced to 3.9 Pa s. In the frequency band 110–150 Hz, the deviation of the theoretical curve from the experimental one was found to be minimal for the layer parameters $\mu = 13.5$ kPa and $\eta = 2.9$ Pa s. The corresponding theoretical dependences obtained for the frequency bands 60–90 and 110–150 Hz are shown by the dashed lines in the insets in Fig. 3. One may conclude that, for the given polymer material, the shear viscosity decreases with increasing frequency while the shear modulus increases. This result agrees well with the experimental results obtained for shear waves in the form of unipolar pulses with durations smaller than 2 ms, which were excited in a similar material by a focused ultrasonic beam [7]. In particular, in [7], the value of the shear viscosity coefficient was found to be (0.30 ± 0.03) Pa s, which is an order of magnitude smaller than the values we obtained in the present study. It should be noted that the frequency spectrum of a shear wave pulse is rather broad (0–800 Hz) and, hence, the value reported in [7] is an average value that mainly characterizes the behavior of shear viscosity at frequencies above 300 Hz.

From the analysis of solution (5) for the acceleration ratio, it follows that the resonance frequencies of

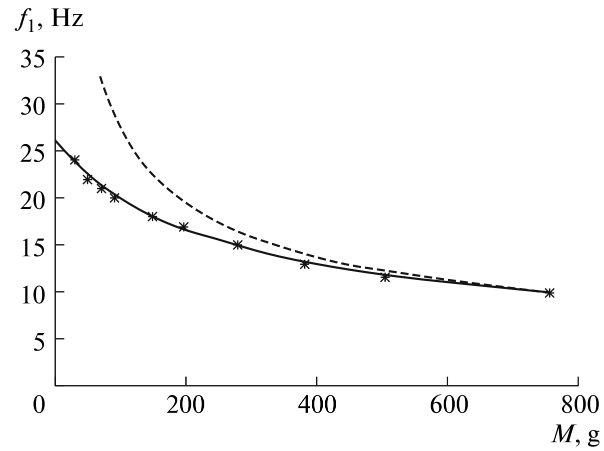

Fig. 4. Dependence of the first resonance frequency on the mass of the upper plate: experiment (dots) and calculation for a layer with the parameters μ = 11.3 kPa and η = 6.2 Pa s (the solid line). The dashed line represents the behavior of the first resonance frequency for large values of the upper plate mass in the absence of shear viscosity.

the elastic layer loaded with a plate decrease as the mass of the upper plate increases. We measured the resonance curves of the elastic layer with additional mass loads of 20–725 g. The experimentally measured dependence of the first resonance frequency of the elastic layer on the mass of the upper plate is shown by dots in Fig. 4. From this dependence, it is possible to determine the value of the shear modulus more accurately. For this purpose, we calculated the theoretical dependences of the first resonance frequency of the layer on the mass of the upper plate by varying the shear modulus from 10 to 12 kPa at a step of 0.1 kPa (at a fixed value of η) and determined the dependence for which the rms deviation from experimental data was minimal. This dependence is shown by the solid line in Fig. 4. As a result, the shear modulus proved to be

$$\mu = 11.3 \pm 0.1\ \text{kPa}.$$

When the plate mass exceeds the layer mass, in the absence of attenuation, the first resonance frequency can be determined from the approximate formula

$$f_1 \approx \frac{c_t}{2\pi}\sqrt{\frac{\rho S}{L}}\frac{1}{\sqrt{M}}. \quad (8)$$

Dependence (8), which is shown in Fig. 4 by a dashed line, agrees well with experimental data for $M > 500$ g.

Figure 5 displays the calculated dependences of the first three resonance frequencies of the loaded elastic layer with fixed values of the shear modulus and shear viscosity (μ = 11.3 kPa and η = 6.2 Pa s) on the mass of the loading plate. As the plate mass increases, the first resonance frequency slowly decreases while the second and third resonance frequencies, which correspond to $n = 2$ and 3, tend to the double and quadruple of the first resonance frequency, respectively. Indeed, as the mass of the upper plate grows, its acceleration decreases at a fixed force acting from the side of the elastic layer. As a result, with increasing plate mass, the upper boundary of the layer becomes less mobile, and, in the limit $M \longrightarrow \infty$, resonances occur at the frequencies corresponding to a layer with a fixed upper boundary: $f_n = \dfrac{c_t n}{2L}$, $n = 1, 2, \ldots$.

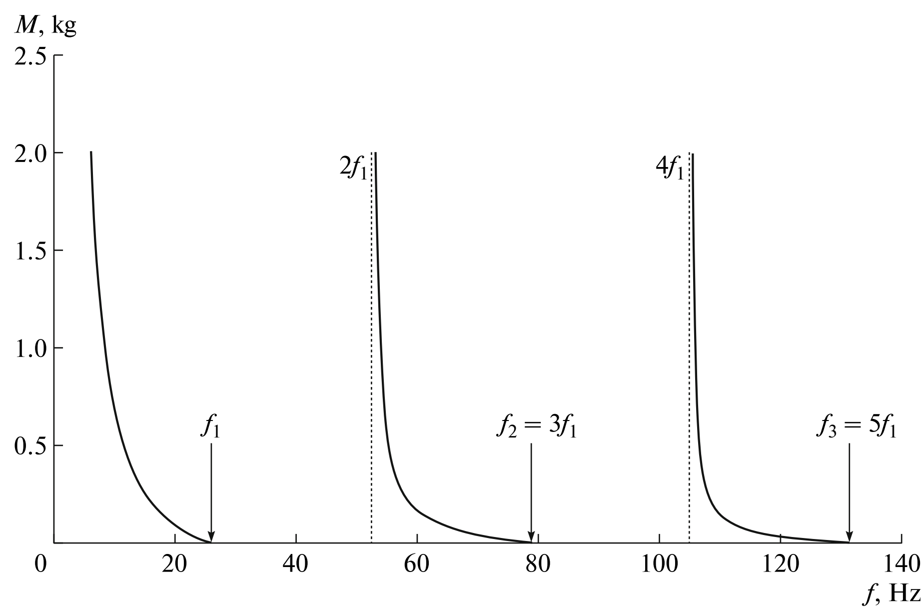

Fig. 5. Calculated dependences of the first three resonance frequencies of the elastic layer on the mass of the upper plate for a layer with the parameters μ = 11.3 kPa and η = 6.2 Pa s.

CONCLUSIONS

Our study shows that the interferometric method allows one to determine the shear modulus and the shear viscosity coefficient at low sonic frequencies with the use of a 1-cm-thick layer of a rubberlike material. The employment of standing waves allows a considerable amplification of the shear displacement amplitude at the resonance frequency. In particular, the gain coefficient at a frequency of 24 Hz was found to be 16; i.e., the resonance properties of the elastic layer were well pronounced.

The proposed method of increasing the family of resonance curves by varying the mass of the upper plate allowed us to improve the accuracy of the shear modulus measurement: the measurement error did not exceed 1%. For comparison, the static measurement error under the same conditions makes 9 and 10% (for the method of static deformation of the layer and the method of pressing a hard ball into the layer, respectively) [7].

We note that the method we used also allows one to study the frequency dependences of the shear modulus $\mu(\omega)$ and shear viscosity coefficient $\eta(\omega)$ in the low-frequency band. It is shown that the shear viscosity coefficient decreases with increasing frequency: at frequencies above 100 Hz, its value is half the value observed at low-frequencies (5–50 Hz). At the same time, the shear modulus and, hence, the shear wave velocity increase with increasing frequency. Such a behavior of elasticity and viscosity is characteristic of a relaxing medium with a relatively long (100 ms) relaxation time [11, 12]. This phenomenon may be related to the mechanism of low-frequency relaxation of shear stresses arising in polymers with long molecules. The interferometric method considered by us allows one to measure in sufficient detail the frequency dependences of viscoelastic parameters of a rubberlike material by varying the resonance frequencies of the layer with the use of additional loading masses. This makes it possible to measure the slow relaxation times and to construct the rheological model of a given rubberlike material.

At low frequencies, the shear viscosity coefficient of the polymer under study was found to be 6.2 ± 1.6 Pa s. This value is an order of magnitude higher than the values obtained for materials made on the basis of gelatin and agar (see, e.g., [13]). Therefore, plastisol is unpromising as a material for the nonlinear mode of operation, when it is necessary to obtain relative shear strains on the order of 1. Our next studies will be devoted to the search for rubberlike materials with a small shear viscosity that are suitable for observation of nonlinear effects in resonators.

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